

# Linear fragments of Peano arithmetic

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## Section 1

# Linear arithmetics

# Linear arithmetic

Recall that:

**Presburger arithmetic** is the full-induction arithmetic for the language  $\langle 0, 1, +, \leq \rangle$ .

**Peano arithmetic** is the full-induction arithmetic for the language  $\langle 0, 1, +, \cdot, \leq \rangle$ .

We introduce:

**Linear arithmetic ( $LA$ )** is the full-induction arithmetic for the language  $\langle 0, 1, +, a \cdot, \leq \rangle$ , where  $a \cdot$  is a **unary function of multiplication** by a positive non-standard element.

## Higher order linear arithmetics

Similarly, for any cardinal  $\kappa$ , we introduce:

$\kappa$ -linear arithmetic ( $LA_\kappa$ ) is the full-induction arithmetic for the language  $\langle 0, 1, -, +, a_\alpha \cdot, \leq \rangle_{\alpha \in \kappa}$ , where all “scalars”  $a_\alpha$  are non-standard.

Then

- $Pr = LA_0$ ,
- $LA = LA_1$ ,
- $LA_2$ ,
- $LA_3$ ,
- $\vdots$

Models of linear arithmetics can be understood as certain **ordered modules**:

Every model  $\mathcal{A} = \langle A, 0, 1, +, a \cdot, \leq \rangle$  of  $LA$  can be canonically “**definably**” extended to a **(discretely ordered) module**

$\mathcal{M}_{\mathcal{A}} = \langle M, 0, 1, +, -, r, \leq \rangle_{r \in R_a}$ , where  $M = A \cup -A$ , over the ring  $R_a = \mathbb{Q}[a] \cap M$ .

For example:

- $(2a) \cdot x$  is defined as  $a \cdot x + a \cdot x$ ,
- $(a^2) \cdot x$  is defined as  $a \cdot (a \cdot x)$ ,
- $(a/2) \cdot x$  is defined as the unique  $y$  such that  $a \cdot x = y + y$  (assuming  $a/2 \in R_a$ ).

More generally, any  $\mathcal{A} = \langle A, 0, 1, +, a_\alpha \cdot, \leq \rangle_{\alpha \in \kappa} \models LA_\kappa$  corresponds to a discretely ordered module  $\mathcal{M}_{\mathcal{A}}$  over the ring  $\mathbb{Q}[a_\alpha]_{\alpha \in \kappa} \cap M$ .

We often identify  $\mathcal{A}$  and  $\mathcal{M}_{\mathcal{A}}$ .

## Section 2

### Context

We want to understand definable sets in models of linear arithmetics (certain discretely ordered modules).

There are two closely related, but simpler situations:

$$\langle M, 0, -, +, r \rangle_{r \in R} \quad \text{and} \quad \langle M, 0, 1, -, +, \leq \rangle = \langle M, 0, 1, -, +, \leq, z \rangle_{z \in \mathbb{Z}}$$

(unordered) modules models of Presburger arithmetics =  
(forget the ordering) = discretely ordered inductive  $\mathbb{Z}$ -modules  
(forget the scalars)



## Unordered modules

The following is a classical result in the theory of modules:

### Theorem (Baur-Monk)

Let  $\mathcal{M} = \langle M, 0, -, +, r \rangle_{r \in R}$  be a module over a ring  $R$  (associative, with 1). Then every formula in  $\mathcal{M}$  is equivalent to a **Boolean combination of primitive positive formulas**, i.e. to a Boolean combination of formulas of the form  $(\exists \bar{z})\psi$ , where  $\psi$  is a **system of linear equations**.

**Remark:** A formula in a language  $L$  is called **primitive positive**, or **pp-formula**, if it is of the form  $(\exists \bar{z}) \bigwedge_{i < n} \chi_i$ , where  $\chi_i$  are atomic formulas.

## Models of $\text{Pr}$

For models of Presburger arithmetic, we have:

### Theorem (Presburger)

Every formula is in  $\text{Pr}$  equivalent to a *disjunction of primitive positive formulas*, i.e. to a formula of the form  $\bigvee_{i < n} (\exists \bar{z}) \psi_i$ , where each  $\psi_i$  is a *system of linear inequalities*.

**Question:** Do the pp-elimination results of Baur-Monk and Presburger generalize to arithmetics  $LA_\kappa$  with  $\kappa > 0$ ?

We “show” that the answer is “Yes” if and only if  $\kappa = 1$ .

Moreover, we completely characterize definable sets in models of  $LA_1$  in the process.

## Section 3

# Definable sets in models of $LA_1$

All the results will be shown in a more general context of certain discretely ordered modules.

Recall that  $\mathcal{M} \models LA$  can be understood as a discretely ordered module over the ring  $R_a = \mathbb{Q}[a] \cap M$ .

We have  $\mathbb{Z}[a] \subseteq R_a \subseteq \mathbb{Q}[a]$  and each  $R_a$  is (up to isomorphism) uniquely determined by the class  $\tau$  of remainders of  $a \bmod n \in \mathbb{N} - \{0\}$ .

$R_\tau$  denotes an  $R_a$  with  $a$  having the remainders  $\tau$ .

It is easy to see that  $\mathcal{M}$  is “integrally-divisible” over  $R_a$ , i.e.  
 $(\exists x, z \in M)(m = rx + z \ \& \ 0 \leq z < r \cdot 1)$  for all  $m \in M$ ,  $r \in R_a$ .

Let further on  $R = R_\tau$  be fixed and  $\mathcal{M} = \langle M, 0, 1, -, +, \leq, r \rangle_{r \in R}$  be a fixed discretely ordered (with 1 being the least positive element), integrally-divisible  $R$ -module.

In particular, any  $\mathcal{M} \models LA$  with  $R = R_a$  satisfy these assumptions.

We show that  $\mathcal{M}$  has **quantifier elimination** in the language extended by functions  $r^{-1}$  **providing integral division by all scalars**  $0 < r \in R$ .

Let  $\mathcal{M}'$  denote the expansion of  $\mathcal{M}$  by definitions of functions  $r^{-1}$ :

$$r^{-1}(x) = y \leftrightarrow (\exists z)(x = ry + z \ \& \ 0 \leq z < r \cdot 1).$$

### Theorem

For any formula  $\varphi(x, \bar{y})$  of  $\mathcal{M}'$  there are *finitely many* terms  $t_i(\bar{y})$ ,  $i < n \in \mathbb{N}$ , of  $\mathcal{M}'$  such that

$$\mathcal{M}' \models (\exists x)\varphi(x, \bar{y}) \leftrightarrow \bigvee_{i < n} \varphi(t_i(\bar{y}), \bar{y}).$$

## Corollary

- 1  $\mathcal{M}'$  has quantifier elimination.
- 2 In  $\mathcal{M}$ , every formula is equivalent to a *disjunction of primitive positive formulas*, i.e. to a formula of the form  $\bigvee_{i < n} (\exists \bar{z}) \psi_i$ , where each  $\psi_i$  is a *system of linear inequalities*.

## Remarks:

- 1 This is a common generalization of QE theorems of Baur-Monk and Presburger.
- 2 The congruences  $\equiv_n$  of the QE-language of  $\text{Pr}$  are quantifier-free definable from the functions  $r^{-1}$ . Indeed,  $x \equiv_n y \leftrightarrow n \cdot n^{-1}(x - y) = x - y$ . (But  $r^{-1}$  is not quantifier-free definable from  $\equiv_r$  for nonstandard  $r$ .)

The proof is quite long and technical. Consists of developing a **calculus** of so called **bracket functions**  $\left[ \begin{smallmatrix} q \\ r \end{smallmatrix} \right] (x) = r^{-1}(qx)$  with  $q, r \in R$ , which extends the calculus of **continued fractions**.

The problem is **much harder** than that of unordered modules or Pr. A simple illustration:

When trying to eliminate the quantifiers in the formula

$$(\exists x, y \geq 0)z = qx + ry,$$

with  $0 < q, r \in R$  fixed, one has to deal with the set

$$S = \{qx + ry; 0 \leq x, y \in M\}.$$

Above  $L = \text{lcm}(q, r)$  we have  $S$  is equal to  $\{px; x \in M\}$  for  $p = \text{gcd}(q, r)$  which is definable by the congruence  $x \equiv_p 0$  (and thus “eliminable”).

But below  $L$ , the set  $S$  is quite **messy**.

$$S = \{qx + ry; 0 \leq x, y \in M\}$$

Above  $L$ :  $S = \{px; x \in M\}$ . Below  $L$ :  $S$  messy.

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Now:

- **For Pr:**  $L$  and thus also  $S \cap [0, L]$  are **finite**  $\Rightarrow$  eliminable,
- **For unordered modules** (Baur-Monk): **No inequalities**, so we have  $S = \{qx + ry; x, y \in M\}$  equal to  $\{px; x \in M\}$  **everywhere**  $\Rightarrow$  eliminable,
- **For  $LA_1$ :**  $S \cap [0, L]$  **messy** and can be **infinite** if  $q, r$  are nonstandard  $\Rightarrow$  a problem...

... and this is just the simplest case. The general problem is **multidimensional**, needs to deal also with inverses  $r^{-1}$ , ...



## Definable functions and sets

Further on, by a term or formula, we mean always an  $\mathcal{M}'$ -term or an  $\mathcal{M}'$ -formula. We will denote by  $\mathbb{C}$  the set of all constant terms in  $\mathcal{M}'$ .

We are going to characterize all **definable functions** and **definable sets** in  $\mathcal{M}$ .

By the QE Theorem, every definable function is a “**piecewise term**”, i.e. an expression of the form

$$\tau(\bar{x}) = \left\{ t_i(\bar{x}) \text{ if } \psi_i(\bar{x}), i < n, \right.$$

where  $t_i$  are terms,  $\psi_i$  are formulas that define a disjoint covering of  $M^{l(\bar{x})}$ .

The terms of  $\mathcal{M}'$  can be **quite complex** as functions  $r^{-1}$  are not distributive w.r.t. addition and do not commute with multiplication; e.g.  $r^{-1}(qx + px)$  cannot be easily simplified.

A term  $t(\bar{x})$  is called **harmonic** if it is a **linear combination of basic “harmonic” terms  $r^{-1}$** :

$$t(\bar{x}) = \sum_{i=0}^{N-1} q_i r_i^{-1}(x_{f(i)}) + c,$$

for some  $q_i, r_i \in D$ ,  $c \in C$  and  $f : N \rightarrow I(\bar{x})$ .

A formula or a piecewise term are harmonic if all its maximal subterms (including those in the side-formulas  $\psi_i$ ) are. They are quantifier-free if all maximal subformulas are quantifier free.

## Harmonic form theorem

A piecewise-term  $\tau$  is called an **almost-term** if it is of the form

$$\tau(\bar{x}) = \left\{ s(\bar{x}) + c_i \text{ if } \psi_i(\bar{x}), i < n, \right.$$

where  $s(\bar{x})$  is a term, and  $c_i \in \mathbb{C}$ , for  $i < n$ .

### Theorem (Harmonic form theorem)

- 1) For every term  $t(\bar{x})$ , there is an **quantifier free harmonic almost-term**  $\tau(\bar{x})$  such that  $\mathcal{M}' \models t(\bar{x}) = \tau(\bar{x})$ .
- 2) For every formula  $\varphi(\bar{x})$ , there is a **quantifier free harmonic formula**  $\psi(\bar{x})$  such that  $\mathcal{M}' \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$ .

## Representation of definable sets

“Every definable set is a finite union of linear images of polyhedra.”

More precisely: For  $\bar{a} \in M$  we denote by  $K(\bar{a})$  the  $l(\bar{a})$ -dimensional box  $K(\bar{a}) = \prod_{i < l(\bar{a})} [0, a_i]$ .

A **polyhedron** over  $X \subseteq M$  is any subset of  $M^m$ , for any  $m$ , definable by a **system of linear inequalities** with parameters from  $X$ .

### Theorem

Every set  $A \subseteq M^n$   **$X$ -definable** in  $\mathcal{M}'$  (for  $X \subseteq M$ ) can be written as

$$A = \bigcup_{i < k} g[P_i],$$

where  $g : (K(\bar{a}) \times M)^n \rightarrow M^n$  is a “linear coordination” of  $M^n$ ,  $\bar{a} \in M^l$ ,  $l \in \mathbb{N}$ , and  $P_i$ , for  $i < k$ , are finitely many **polyhedra in  $(K(\bar{a}) \times M)^n$  over parameters from  $X$** .

## Section 4

# Properties of $LA$

## Theorem (Properties of $LA$ )

1)  $LA$  is *model-complete*.

Moreover: Every formula is in  $LA$  equivalent to a *disjunction of primitive positive formulas*, i.e. to a formula of the form  $\bigvee_{i < n} (\exists \bar{z}) \psi_i$ , where each  $\psi_i$  is a *system of linear inequalities*.

2) For  $\mathcal{A}, \mathcal{B} \models LA$ , it is  $\mathcal{A} \equiv \mathcal{B} \Leftrightarrow a^{\mathcal{A}} \equiv a^{\mathcal{B}} \pmod{n}$ , for all  $0 < n \in \mathbb{N}$ .

$LA_\tau = LA + \{a \equiv \tau(p, k) \pmod{p^k}; p \in \mathbb{N} \text{ prime}, k \in \mathbb{N}\}$ , for  $\tau \in \prod_{p \in \mathbb{P}} \mathbb{J}_p$ , are *all simple complete extensions* of  $LA$ .

3)  $R_\tau = \{r(a)/n; 0 \leq r(a) \in \mathbb{Z}[a], 0 \neq n \in \mathbb{N}, p^k | r(\tau(p, k)) \forall p^k | n\}$  is the unique *prime model* of  $LA_\tau$ , for  $\tau \in \prod_{p \in \mathbb{P}} \mathbb{J}_p$ .

4)  $LA$  is *decidable*.

$LA_\tau$  is decidable if and only if  $\tau$  is recursive.

5) The *induction scheme* in  $LA$  may be equivalently replaced by the scheme of *integral divisibility* (over certain basic  $LA^-$ )

$(\exists y, z)(x = qy + z \ \& \ z < q)$ , for all  $0 < q \in \mathbb{Z}[a]$ .

**Remark:** All these properties are similar to those well-known for  $Pr$ . But we have already seen that the proof is much harder.

### Corollary

*Up to elementary equivalence, models of  $LA$  are exactly all ultraproducts*

$$\mathbb{N}_{\mathcal{U}} = \left( \prod_{n \in \mathbb{N}} \langle \mathbb{N}, 0, 1, +, n \cdot, \leq \rangle \right) / \mathcal{U},$$

*where  $\mathcal{U}$  is a non-principal ultrafilter on  $\mathbb{N}$ , i.e.  $\mathcal{U} \in \beta\mathbb{N} - \mathbb{N}$ .*

Thus models of  $LA$  can be seen as “pseudofinite”  $\mathbb{Z}$ -modules.

A theory  $T$  is **NIP** (not independence property) if the following **does not** exist in any  $\mathcal{M} \models T$ :

an  $L(T)$ -formula  $\varphi(\bar{x}, \bar{y})$ , sequences  $(\bar{a}_i)_{i \in \mathbb{N}}$ ,  $(\bar{b}_j)_{j \in \mathbb{N}}$ ,

such that  $\mathcal{M} \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \in J$ .

I.e.  $T$  is NIP if **no model of  $T$  encodes the powerset  $\mathcal{P}(\mathbb{N})$  of  $\mathbb{N}$ .**

A theory  $T$  is **dp-minimal** if the following **does not** exist in any  $\mathcal{M} \models T$ :

$L(T)$ -formulas  $\varphi(x, \bar{y})$ ,  $\psi(x, \bar{y})$  sequences  $(\bar{a}_i)_{i \in \mathbb{N}}$ ,  $(\bar{b}_j)_{j \in \mathbb{N}}$ ,

such that for all  $i, j \in \mathbb{N}$  the following has a solution  $x \in M$ :

$\varphi(x, \bar{a}_i) \ \& \ \psi(x, \bar{b}_j) \ \& \ \bigwedge_{k \neq i} \neg \varphi(x, \bar{a}_k) \ \& \ \bigwedge_{l \neq j} \neg \psi(x, \bar{b}_l)$ .



Recall that Presburger arithmetic is dp-minimal and thus also NIP. For  $LA$ , this is different:

### Theorem

*$LA$  is NIP but not dp-minimal.*

**Proof sketch:** For the NIP property use the description of definable sets (in all dimensions) as linear images of finite unions of polyhedra, i.e. solutions of systems of linear inequalities. But linear inequalities have VC-dimension 1 hence are NIP.

But  $LA$  cannot be dp-minimal because  $x \mapsto (x \operatorname{div} a, x \operatorname{mod} a)$  is a definable function which is a bijection between infinite sets  $[0, a^2)$  and  $[0, a)^2$ .

## Application: (In)dependence of Peano multiplications

The following corollary is on the structure of models of Peano arithmetic:

Over a fixed saturated model of  $Pr$ , the value of a saturated “Peano multiplication” in the point  $(c, d)$  is **uniquely determined** by its values on the **line**  $\{(a, x); x \in M\} \subseteq M^2$  if and only if  $c$  or  $d$  is a polynomial in  $a$  over  $\mathbb{Q}$ .

### Corollary

Let  $\mathcal{M} = \langle M, 0, 1, +, \cdot, \leq \rangle$  be a *saturated model of Peano arithmetic*,  $0 \leq a \in M - \mathbb{N}$ ,  $c, d \in M$ . Then the following are equivalent:

- 1)  $c \cdot d = c \circ d$  for every Peano multiplication  $\circ$  on  $\mathcal{M}$  with  $a \cdot x = a \circ x$  for all  $x \in M$ .
- 2)  $c \in \mathbb{Q}[a]$  or  $d \in \mathbb{Q}[a]$ .

## Section 5

# Properties of $LA_2$ and above

## Wild models of $LA_2$

Unlike models of Pr and  $LA$ , models of  $LA_2$  can be model-theoretically wild. There is a model of  $LA_2$  where a **nonstandard initial segment of a Peano multiplication is definable**:

### Proposition (with P. Pudlák)

*There is a model  $\mathcal{M} = \langle M, 0, 1, +, a \cdot, b \cdot, \leq \rangle \models LA_2$  and a nonstandard  $l \in M$  such that  $\cdot \upharpoonright [0, l]^2$  is definable in  $\mathcal{M}$  for some Peano multiplication  $\cdot$  on  $\mathcal{M}$  (i.e. such that  $\langle M, 0, 1, +, \cdot, \leq \rangle \models PA$ ).*

Note that no such model can exist for  $LA_1$  (easy consequence of model-completeness).

Unlike Pr and  $LA$ :

### Corollary

*For  $\kappa \geq 2$ , the theory  $LA_\kappa$  does not have pp-elimination. It is not even model complete (i.e. does not have elimination to  $\exists$ -formulas).*

## Proof idea:

Multiplication on  $[0, l]^2$  can be defined from the **squaring function** on  $[0, 2l]$  by  $x \cdot y = ((x + y)^2 - x^2 - y^2)/2$ .

In a saturated model of Peano arithmetic, for any  $\mathbb{N} < l \in M$ , we find elements  $a, b$  such that the sequence  $(1, 1^2, 2, 2^2, \dots, 2l, (2l)^2)$  is encoded by the set of all **numerators of convergents of the continued fraction of  $a/b$** . The crucial observation is that this set is definable in the language of  $LA_2$  with  $a, b$  as the two scalars.

(Note that for  $LA_1$  every continued fraction of scalars is finite, thus the construction above does not allow to define an infinite part of a Peano multiplication.)

The following is well known:

### Proposition

*(Unordered) modules are stable and thus NIP.  
Presburger arithmetic (though unstable) is NIP.*

**Question (Chernikov and Hils):** Is every ordered module NIP?

We answer **NO**:

### Corollary (with P. Pudlák)

*The wild model  $\mathcal{M}$  of  $LA_2$  from the previous proposition, considered as an ordered module over  $\mathbb{Z}[a, b] / =_{\mathcal{M}}$ , is not NIP.*

We know that **no pp-elimination** is possible for the arithmetics  $LA_\kappa$  with  $\kappa \geq 2$ . However, the arithmetics  $LA_\kappa$  satisfy **bounded QE for all  $\kappa$** :

Recall that an ordered  $R$ -module  $\mathcal{M} = \langle M, 0, 1, +, -, \leq, r \rangle_{r \in R}$  with a unit  $1 > 0$  is called integrally divisible if

$\mathcal{M} \models (\forall x)(\exists y, z)(x = ry + z \ \& \ 0 \leq z < r \cdot 1)$  for every  $0 < r \in R$ .

### Theorem

*Let  $\mathcal{M}$  be an integrally divisible ordered module with unit. Then every formula is in  $\mathcal{M}$  equivalent to a bounded formula.*

### Corollary

*For any  $\kappa$ , every  $\mathcal{M} \models LA_\kappa$  has bounded quantifier elimination.*

Hence, **in no model of  $LA_\kappa$  a Peano multiplication is definable on the whole universe.**

## Proof idea:

The crucial part is to prove that in  $\mathcal{M}$ , every formula  $\psi(\bar{x})$  is **protoperiodic**.

$\psi(\bar{x})$  is protoperiodic if there exists a disjoint covering  $\mathcal{A}$  of  $M^{l(\bar{x})}$  by finitely many convex polyhedra, such that for any  $A \in \mathcal{A}$  and a direction  $s = \bar{\alpha} \in R^{l(\bar{x})}$  either  $A$  is bounded in the direction  $s$  ( $\exists 0 \leq m \in M$  s.t.  $A + sm \not\subseteq A$ ) or there is a period  $0 < P \in R$  for  $\psi$  on  $A$  in the direction  $s$  (i.e.  $(\forall \bar{u}, \bar{v} \in A)((\exists m \in M)\bar{u} = Psm + \bar{v} \rightarrow (\psi(\bar{u}) \leftrightarrow \psi(\bar{v})))$ ).

This is done by induction on complexity of  $\psi$ .



## Tame and wild linear fragments of $PA$

Let  $\mathcal{M} \models PA$  and let  $R$  be a subring of  $\mathcal{M} \cup -\mathcal{M}$ . Denote  $M_R$  the  $R$ -(semi)module fragment of  $\mathcal{M}$ . We know the following:

### Proposition

*If  $R$  is some  $R_\tau$  then  $M_R$  has pp-elimination (hence is model complete).*

But we can also prove the following:

### Proposition

*If  $R$  contains elements  $a, b$  such that  $a/b$  has an infinite continued fraction (in  $\mathcal{M}$ ) then  $M_R$  is not model complete (hence does not have pp-elimination).*

**Proof sketch:** The set  $C$  of all pairs  $(u, v)$  such that  $u/v$  is a convergent of  $a/b$  (in the lowest terms) is infinite and definable in  $M_R$ . It is easy to show that  $C$  is a (graph of a) **concave sequence**. But then it cannot be defined by an  $\exists$ -formula, as each infinite  $\exists$ -definable set in  $M_R$  contains three points lying on a line.

## Open questions and Thank you

### Open questions:

Is it possible to decide pp-elimination/model completeness of  $M_R$  also in the remaining cases (when  $R$  is not  $R_\tau$  but does not contain  $a, b$  with an infinite continued fraction)? Do such cases exist at all?

Is there a model  $M_R$  which is not model complete but is still NIP?

What about full-induction modules which are not fragments of models of PA?

In particular, is there a non-trivial (i.e. with  $R = \mathbb{Z}[a, b] \cap M$  not any  $R_\tau$ ) model of  $LA_2$  which has pp-elimination/is model complete/is NIP?

Can the bounded quantifier elimination for models of  $LA_\kappa$  with  $\kappa \geq 2$  be strengthened? If yes, how does it depend on  $\kappa$ ?

Thank you for your attention.

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