Shepherdson's theorems for fragments of open induction

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Open induction

The open induction arithmetic (IOpen) is the formal arithmetical theory in the language $L = \langle 0, 1, +, -, \cdot, \leq \rangle$ axiomatized by a set of elementary properties of $0, 1, +, -, \cdot, \leq$ and by the scheme of axioms of open induction:

$$(\varphi(0, \overline{y}) \,\&\, (\forall x)(\varphi(x, \overline{y}) \rightarrow \varphi(x+1, \overline{y})) \rightarrow (\forall x)\varphi(x, \overline{y})),$$

for all open (i.e. quantifier free) formulas φ of L.

The axioms ensure that any model $\mathcal{M}^+$ of IOpen is the nonnegative part of the unique discretely ordered ring $\mathcal{M}$.

We often say that a discretely ordered ring $\mathcal{M}$ is a model of IOpen, by which we mean that it's nonnegative part $\mathcal{M}^+$ is.

The Shepherdson's theorem

Question: Is there an algebraic characterisation of those discretely ordered rings that are models of IOpen?

Theorem (J. C. Shepherdson, 1964)

A discretely ordered ring is a model of IOpen *if and only if it is an integer part of it's real closure.*

The real closure $\mathrm{R}(\mathcal{M})$ of a discretely ordered ring $\mathcal{M}$ is the unique (ordered) real closed field extending the ordered fraction field $\mathrm{F}(\mathcal{M})$ of $\mathcal{M}$.

A discretely ordered subring I of $\mathcal{R}$ is called an integer part of $\mathcal{R}$ (denoted by $I \subseteq^{IP} \mathcal{R}$) if for every $r \in R$, there is $i \in I$ such that $r - 1 < i \leq r$. Such an i is called the integer part of r (w.r.t. I).

Integer-parts-of-roots property

Let $f(x)$ be a definable unary function on $\mathcal{M}$. By $\mathrm{IPR}(f)$ (integer-parts-of-roots) we denote the following formula:

$$(a < b \,\&\, f(a) \leq y < f(b)) \rightarrow (\exists x)(a \leq x < b \,\&\, f(x) \leq y < f(x+1)).$$

The intended meaning of $\mathcal{M} \models \mathrm{IPR}(f)$ can be expressed in vague terms as "existence of integer parts for all f-roots of values $y \in M$".

If $\mathcal{F}$ is a set of definable unary functions on $\mathcal{M}$, we write $\mathrm{IPR}(\mathcal{F})$ for the scheme $\{\mathrm{IPR}(f); f \in \mathcal{F}\}$.

The Shepherdson's theorem reformulated

We can stratify Shepherdson's theorem a litle bit more:

Theorem (J. C. Shepherdson, 1964)

The following are equivalent for any discretely ordered ring $\mathcal{M}$:

1. $\mathcal{M} \subseteq^{IP} R(\mathcal{M})$,
2. *all roots $r \in R(\mathcal{M})$ of polynomials $p \in M[x]$ have integer parts in M,*
3. $\mathcal{M} \models \mathrm{IPR}(M[x])$,
4. $\mathcal{M}^+ \models \mathrm{IOpen}$.

Presburger arithmetic

The Presburger arithmetic (Pr) is the formal arithmetical theory in the language $L^+ = \langle 0, 1, +, -, \leq \rangle$ axiomatized by a set of elementary properties of $0, 1, +, -, \leq$ and by the scheme of axioms of induction:

$$(\varphi(0, \overline{y}) \,\&\, (\forall x)(\varphi(x, \overline{y}) \rightarrow \varphi(x+1, \overline{y})) \rightarrow (\forall x)\varphi(x, \overline{y})),$$

for all formulas φ of L^+.

Every model of Pr is the nonnegative part of the unique discretely ordered additive Abelian group.

“Shepherdson’s theorem” for Presburger arithmetic

Theorem

Let $\mathcal{M}$ be a discretely ordered additive Abelian group. Then the following are equivalent:

1. $\mathcal{M} \subseteq^{IP} \frac{M}{\mathbb{N}}$,
2. *all fractions* m/n *with* $m \in M$ *and* $0 \neq n \in \mathbb{N}$ *have integer parts in* M,
3. $\mathcal{M} \models \mathrm{IPR}(\{n(x); n \in \mathbb{N}\})$, *where* $n(x) = x + \ldots + x$ *with* n *summands,*
4. $\mathcal{M}^+ \models \mathrm{Pr}$.

The Shepherdson's theorem reformulated

Let us recall the Shepherdson's theorem for IOpen:

Theorem (J. C. Shepherdson, 1964)

The following are equivalent for any discretely ordered ring $\mathcal{M}$:

1. $\mathcal{M} \subseteq^{IP} R(\mathcal{M})$,
2. *all roots $r \in R(\mathcal{M})$ of polynomials $p \in M[x]$ have integer parts in M,*
3. $\mathcal{M} \models \mathrm{IPR}(M[x])$,
4. $\mathcal{M}^+ \models \mathrm{IOpen}$.

The property $\mathcal{M} \subseteq^{IP} \mathrm{R}(\mathcal{M})$ is equivalent to

$$\mathcal{M} \subseteq^{IP} \mathrm{F}(\mathcal{M}) \,\&\, \mathrm{F}(M) \subseteq^{d} \mathrm{R}(\mathcal{M}),$$

where $\mathcal{R}' \subseteq^{d} \mathcal{R}$ denotes that R' is a dense subring of R, i.e. for every $q < r$ from R there is $r' \in R'$ such that $q < r' < r$.

Open linear induction

The theory corresponding (in a Shepherdson-like way) to the property $\mathcal{M} \subseteq^{IP} \mathrm{F}(\mathcal{M})$ is the the open linear induction arithmetic.

The open linear induction arithmetic ($\mathrm{IOpenLin}$) is the formal arithmetical theory in the language $L = \langle 0, 1, +, -, \cdot, \leq \rangle$ axiomatized by a set of elementary properties of $0, 1, +, -, \cdot, \leq$ and by the scheme of axioms of open linear induction:

$$(\varphi(0, \overline{y}) \,\&\, (\forall x)(\varphi(x, \overline{y}) \rightarrow \varphi(x+1, \overline{y})) \rightarrow (\forall x)\varphi(x, \overline{y})),$$

for all open (i.e. quantifier free) linear formulas φ of L.

The formula $\varphi(x, \overline{y})$ is called linear if in every occurrence of $\cdot$ in φ at least one of the two factors is y_i, for some i.

Shepherdson theorem for $\mathrm{IOpenLin}$

Theorem

Let $\mathcal{M}$ be a discretely ordered ring. Then the following are equivalent:

1. $\mathcal{M} \subseteq^{IP} F(\mathcal{M})$,
2. *all fractions* m'/m *with* $m', m \in M$ *and* $0 < m$ *have integer parts in* M,
3. $\mathcal{M} \models \mathrm{IPR}(\{m(x); m \in M\})$, *where* $m(x) = m \cdot x$,
4. $\mathcal{M} \models (\forall n, k \neq 0)(\exists n')(\exists 0 \leq l < k)(n = n' \cdot k + l)$,
5. $\mathcal{M}^+ \models \mathrm{IOpenLin}$.

Open eventual induction

There is a theory that corresponds to the property $\mathsf{F}(M) \subseteq^d \mathsf{R}(\mathcal{M})$ as well. However, it is not given by a scheme of induction, but by certain generalization of it - the scheme of open eventual induction.

The l-step induction axiom for the formula $\varphi(x, \bar{y})$ is the following formula denoted by $I_x^l \varphi(x, \bar{y})$ or simply $I_x^l \varphi$:

$$((\forall u < l)\varphi(u, \bar{y}) \wedge (\forall v)(\varphi(v, \bar{y}) \to \varphi(v + l, \bar{y}))) \to (\forall x)\varphi(x, \bar{y}).$$

The open eventual induction arithmetic (eIOpen) is the formal arithmetical theory in the language $L = \langle 0, 1, +, -, \cdot, \leq \rangle$ axiomatized by a set of elementary properties of $0, 1, +, -, \cdot, \leq$ and by the scheme of axioms of open eventual induction:

$$(\exists l > 0)(I_x^l \varphi(x/l, \bar{y})),$$

for all open (i.e. quantifier free) formulas φ of L.

Shepherdson theorem for eIOpen

Theorem

Let $\mathcal{M}$ be a discretely ordered ring. Then the following are equivalent:

1. $F(\mathcal{M}) \subseteq^d R(\mathcal{M})$,
2. $(\forall r \in R(\mathcal{M}))(\exists f \in F(\mathcal{M}))(r < f < r + 1)$,
3. $(\forall r \in R(\mathcal{M}))(\exists l, m \in M)(m \leq lr < m + l)$,
4. $\mathcal{M}^+ \models \mathrm{eIOpen}$.

Alternative axiomatization of IOpen

As we know that

$$(\mathcal{M} \subseteq^{IP} \mathsf{F}(\mathcal{M}) \,\&\, \mathsf{F}(\mathcal{M}) \subseteq^{d} \mathsf{R}(\mathcal{M})) \to \mathcal{M} \subseteq^{IP} \mathsf{R}(\mathcal{M}),$$

we get the following alternative axiomatization of IOpen:

Corollary

$\mathrm{IOpen} = \mathrm{IOpenLin} + \mathrm{eIOpen}.$

Note that similarily from the inclusion

$$\mathrm{IOpen} \supseteq \mathrm{IOpenLin} + \mathrm{eIOpen},$$

we get:

Corollary

$(\mathcal{M} \subseteq^{IP} F(\mathcal{M}) \,\&\, F(\mathcal{M}) \subseteq^{d} R(\mathcal{M})) \leftrightarrow \mathcal{M} \subseteq^{IP} R(\mathcal{M}).$

Open questions, Thank you and references

Open questions:

Is there a truly general Shepherdson's theorem, i.e. a theorem such that all variants of Shepherdson theorems for various theories are its special cases?

Are there variants of the Shepherdson's theorem for theories stronger than IOpen?

Thank you for your attention.

References: *[J. Glivická and P. Glivický, Shepherdson's theorems for fragments of open induction, APLIMAT 2017 proceedings, 583–589, arXiv: 1701.02001.]*