Lifting of Monochromatic Paths in Graph Products and Applications in Model Theory

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Peano products

The presented result is a combinatorial core of some undefinability proofs in model theory. We use it in our study of Peano products.

Proposition (GCH)

Let $\mathcal{M}_+$ be a saturated model of Presburger arithmetic (Pr) of cardinality 2^ω. For any consistent extension T of the Peano arithmetic (PA) there is a product $\cdot$ on M such that $\langle \mathcal{M}_+, \cdot \rangle \models T$.

We investigate the system of all products $\cdot$ on a given $\mathcal{M}_+ \models \mathrm{Pr}$ such that $\langle \mathcal{M}_+, \cdot \rangle \models \mathrm{PA}$. We call these products Peano products on $\mathcal{M}_+$.

From any Peano product we can define the parabola ($x^2 = x \cdot x$) and vice versa ($x \cdot y = \frac{(x+y)^2 - x^2 - y^2}{2}$). We speak about Peano parabolas.

Models of Peano arithmetic
Meeting pairs

We are interested in the system of all Peano products (or parabolas) on a given $\mathcal{M}_+ \models \mathrm{Pr}$. In particular we would like to solve the fundamental question of existence of meeting pairs:

Definition

We say the couple $\langle ()^2, {}^2() \rangle$ of Peano parabolas on $\mathcal{M}_+$ is a meeting pair with a meeting point $a \in M$ if $a^2 = {}^2a$ and there is $b < a$ such that $b^2 \neq {}^2b$.
We say the couple $\langle \cdot, \times \rangle$ of Peano products on $\mathcal{M}_+$ is a meeting pair with a meeting point $a \in M$ if for all $x \in M$ $a \cdot x = a \times x$ and there are $c, d < a$ such that $c \cdot d \neq c \times d$.

Models of Peano arithmetic

Undefinability of short parabola

It can be shown that the problem of existence of meeting pairs can be reduced to an undefinability problem.

Theorem

Let $()^2$ be a Peano parabola on $\mathcal{M}_+$, $a \in \mathcal{M}$ nonstandard. If the short parabola $()^2|_{[0,a-1]}$ isn't piecewise definable in $\langle \mathcal{M}_+, a, a^2 \rangle$ then there exists a Peano parabola ${}^2()$ such that $\langle ()^2, {}^2() \rangle$ is a meeting pair with the meeting point a.

Theorem

Let $\cdot$ be a Peano product on $\mathcal{M}_+$, $()^2$ its parabola, $a \in \mathcal{M}$ nonstandard. If the short parabola $()^2|_{[0,a-1]}$ isn't piecewise definable in $\langle \mathcal{M}_+, a \cdot \_ \rangle$ then there exists a Peano product $\times$ such that $\langle \cdot, \times \rangle$ is a meeting pair with the meeting point a.

Monochromatic paths theorem

Abstract

We examine cartesian products $\prod_{\beta \in I} \mathcal{G}_\beta$ of oriented edge-colored graphs $\mathcal{G}_\beta$ with particular regard to existence of monochromatic paths in subsets of these products. Under some assumptions on $\mathcal{G}_\beta$'s given a set $B \subseteq \prod_{\beta \in I} \mathcal{G}_\beta$ such that its projection $\pi_{I,J}(B)$ contains a long monochromatic path we provide a sufficient descriptive condition for B to contain monochromatic path of length n.

Our proof uses Szemerédi's theorem.

$\mathcal{G} = \langle V, E, C\rangle$ denotes a fixed oriented edge-colored graph (possibly with loops and/or multiple edges); V a set of its vetices, E an edge relation, C a family of colours c_i.

Definition

We define relations $\rightarrow_d, \stackrel{c}{\rightarrow}_d$ on V for all $d \in \omega \setminus \{0\}$, $c \in C$ in the following way:
$x \rightarrow_d y \Leftrightarrow$ there is a monochromatic path of length d from x to y
$x \stackrel{c}{\rightarrow}_d y \Leftrightarrow$ there is a path of length d and colour c from x to y.

Definition

For $d \in \omega \setminus \{0\}$ we define d'th derivation of $\mathcal{G}$ to be the graph $\mathcal{G}^{(d)} = \langle V^{\mathcal{G}}, \rightarrow_d^{\mathcal{G}}, C^{\mathcal{G}}/_{\rightarrow_d^{\mathcal{G}}}\rangle$.

Definition

Let $n \in \omega \setminus \{0\}$. We say $\mathcal{G}$ is semitransitive if for all $0 < d \in \omega$ and each colour $c \in C$ there is a colour $c_d \in C$ such that for all $x, y \in V$ if $x \stackrel{c}{\to}_d y$ then there is edge from x to y of colour c_d.

Definition

Let $\mathcal{H}$ be an oriented graph, κ a cardinal. $\mathcal{H}$ is called $<\kappa$-complete if it is possible to divide $V^{\mathcal{H}}$ to $< \kappa$ sets V_α such that each $V_\alpha, E^{\mathcal{H}}|_{V_\alpha}$ is a complete oriented graph.

Definition

Let $n \in \omega \setminus \{0\}$ *and* $X \subseteq V$. *We say that* X *is n-convex if whenever* σ *is a monochromatic path of length n in* $\mathcal{G}$ *such that the first and the last vertex of* σ *lie in* X *then the whole* σ *lies in* X.

Definition

The (monochromatic) length of the set $X \subseteq V$ *is defined as follows:*

$$l(X) = sup\,\{n \in \omega;\ (\exists x, y \in X)(x \rightarrow_n y)\}$$

Let $\langle \mathcal{G}_\beta; \beta \in \mathbb{I}\rangle$ be a system of oriented edge-colored graphs where $\mathcal{G}_\beta = \langle V_\beta, E_\beta, C_\beta\rangle$.
For $J \subseteq I \subseteq \mathbb{I}$ there is the projection $\pi_{I,J} : \prod_{\beta\in I} \mathcal{G}_\beta \to \prod_{\beta\in J} \mathcal{G}_\beta$ which forgets the coordinates $I \setminus J$.
Also for any bijection $\gamma : I \to I'$ such that $\mathcal{G}_i = \mathcal{G}_{\gamma(i)}$ for all $i \in I$ there is the change of coordinates $\overline{\gamma} : \prod_{\beta\in I} V_\beta \to \prod_{\beta\in I'} V_\beta$.

Definition

Let $X \subseteq Y$ be subsets of $\prod_{\beta\in I} V_\beta$. Then X is called J-subset of Y (denoted $X \subseteq_J Y$) if $(\pi_{I,J}^{-1} \circ \pi_{I,J})(X) \cap Y = X$.

Definition

Let $\mathcal{S} \subseteq \bigcup_{I\subseteq\mathbb{I}} \mathcal{P}(\prod_{\beta\in I} V_\beta)$. By $\overline{\mathcal{S}}$ we denote the least system of elements of $\bigcup_{I\subseteq\mathbb{I}} \mathcal{P}(\prod_{\beta\in I} V_\beta)$ which contains $\mathcal{S}$ and is closed under: finite unions, finite intersections, projections $\pi_{K,L}$ where $L \subseteq_{cofin} K \subseteq \mathbb{I}$, preimages $\pi_{K,L}^{-1}$ where $L \subseteq K \subseteq \mathbb{I}$ and changes of coordinates. The system of all finite unions of J-subsets of elements of $\overline{\mathcal{S}} \cap \prod_{\beta\in I} V_\beta$ is denoted $\overline{\mathcal{S}}^I_J$.

Lemma (normal form)

All $B \in \overline{\mathcal{S}}$ can be expressed in the (so called) normal form:

$$B = \bigcup_{i=1}^{u} \pi_{K_i,I} \left(\bigcap_{j} T_{i,j} \right)$$

where $T_{i,j} = (\pi^{-1}_{K_i,L_{i,j}} \circ \gamma)(S_{i,j})$, γ is the change of coordinates and $S_{i,j} \in \mathcal{S}$ and $I \subseteq_{cofin} K_i \subseteq \mathbb{I}'$ where $\mathbb{I}' \supseteq \mathbb{I}$ is such that all changes of coordinates used in the expression are inside $\mathbb{I}'$.

Lemma

Let $X, Y \subseteq \prod_{\beta \in I} V_\beta$ be n-convex sets, $\pi_{K,I}$ a projection, γ a change of coordinates. Then all the preimage $\pi^{-1}_{K,I}(X)$, $\gamma(X)$ and $X \cap Y$ are n-convex.

Theorem (monochromatic paths theorem)

Let $\mathbb{I}$ be a set, $J \subseteq_{cofin} I$ subsets of $\mathbb{I}$ and $\mathcal{G}_\beta = \langle V_\beta, E_\beta, C_\beta \rangle$, $\beta \in \mathbb{I}$, be oriented edge-colored graphs. Let $0 < n \in \omega$ be such that the n'th derivations $\mathcal{G}_\beta^{(n)}$ for $\beta \in \mathbb{I}$ are $<\omega$-complete and $\prod_{\beta \in J} \mathcal{G}_\beta$ is n-semitransitive.

If $\mathcal{S}$ is a system of n-convex elements of $\bigcup_{K \subseteq \mathbb{I}} \mathcal{P}(\prod_{\beta \in K} V_\beta)$ then for any set $B \in \overline{\mathcal{S}}_J^I$ if $\pi_{I,J}(B)$ has length in $\prod_{\beta \in J} \mathcal{G}_\beta$ greater than N_B^n then B has length in $\prod_{\beta \in I} \mathcal{G}_\beta$ at least n.

Here N_B^n is a big finite number which can be computed from n and the normal form of B.

Idea of proof: Take the monochromatic path of length N_B^n and divide its vertices into finitely many invariant classes.The invariant is chosen in such a way that each monochromatic path in $\pi_{I,J}(B)$ vertices of which have the same invariant (invariant path) can be lifted to B.Take one of these classes C which is big enough.Then by Szemerédi's theorem and semitransitivity there is a monochromatic invariant path of length n in C.

Connections to definability

Lemma

Let $\mathcal{A}$ be a structure with the support A. Each set $\mathcal{S}$ of formulas can be identified with the system of subsets of $\bigcup_{0<i<\omega} \mathcal{P}(A^i)$ definable by formulas from $\mathcal{S}$.
Then the system $\overline{\mathcal{S}}$ consists of (the sets definable by) the disjunctions of positive primitive formulas ($\vee\mathrm{pp}$-formulas) over $\mathcal{S}$.
Moreover $\overline{\mathcal{S}}^i_J$ are the J-piecewise $\vee\mathrm{pp}$-definable from $\mathcal{S}$ subsets of $\mathcal{P}(A^i)$.

Connections to definability

Let A be a set, $\mathcal{F}$ a system of partial unary functions on A. Then we can define the oriented edge-colored graph $\mathcal{G}^{\mathcal{F},A}$ as follows: the set of vertices is $V^{\mathcal{G}^{\mathcal{F},A}} = A$, the family of colors $C^{\mathcal{G}^{\mathcal{F},A}} = \mathcal{F}$ and there is an oriented edge of color $F \in \mathcal{F}$ from a to b if $F(a) = b$.

The monochromatic paths in $\mathcal{G}^{\mathcal{F},A}$ are exactly $\mathcal{F}$ iterations:

Definition

Let $\mathcal{F}$ be a system of unary functions on A. The sequence $(a_i)_{i=0}^{n-1}$ is called $\mathcal{F}$-iteration of length n if there exists $F \in \mathcal{F}$ such that for all $i < n - 1$ $a_{i+1} = F(a_i)$.

Example

Let $A = \mathbb{N}$, $\mathcal{F}_{ar} = \{F_d;\, d \in \mathbb{Z}\}$ where $F_d(m) = m + d$ whenever $m + d \in \mathbb{N}$.
Then $\mathcal{F}_{ar}$-iterations are just arithmetical progressions and $x \to_n y \Leftrightarrow x \equiv_n y$.
Moreover it can be easily verified that $\mathcal{G}^{\mathcal{F},A}$ satisfies all assumptions from the Monochromatic paths theorem.

Example

In the Presburger's arithmetic (Pr) (in the language $0, S, +$) every formula is equivalent to some $\vee\mathrm{pp}$-formula. Moreover sets definable by atomic formulas (linear equtions) are clearly n-convex for any n.
For any model $\mathcal{A}$ of Pr one can do the similar reasoning as in the previous example.

Proposition

Let $\mathcal{A} \models \mathrm{Pr}$ and $D \subseteq A^m$ contains an infinite arithmetical progression. If f is a partial m-ary function on A which is defined on D and everywhere n-nonlinear (graph of f doesn't contain any arithmetical progression of length n) then f isn't piecewise definable in $\mathcal{A}$.

Corollary

Let $()^2$ be a Peano parabola on $\mathcal{M}_+$, $a \in \mathcal{M}$ nonstandard. The short parabola $()^2|_{[0,a-1]}$ isn't piecewise definable in $\langle \mathcal{M}_+, a, a^2 \rangle$.

Corollary

Let $()^2$ be a Peano parabola on $\mathcal{M}_+$, $a \in \mathcal{M}$ nonstandard. There is a Peano parabola ${}^2()$ on M such that $\langle ()^2, {}^2() \rangle$ is a meeting pair with the meeting point a.

We can proceed in the completely same way in the question of existence of meeting pairs of Peano products.

In order to do that we need that every formula in LA is equivalent to some $\vee\mathrm{pp}$-formula.This is still an unsolved problem.

Thanks

Thank you for your attention.