

# Model theory of certain discretely ordered modules and linear arithmetics

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## Section 1

### Background

## Baur-Monk Theorem

The following is a classical result in the theory of modules:

### Theorem (Baur-Monk)

Let  $\mathcal{M} = \langle M, 0, -, +, r \rangle_{r \in R}$  be a (left) module over a ring (associative, with 1)  $R$ .

Every formula in  $\mathcal{M}$  is equivalent to a boolean combination of primitive positive formulas, i.e. to a boolean combination of formulas of the form  $(\exists \bar{z})\psi_i$ , where each  $\psi_i$  is a system of linear equations.

**Remark:** A formula in a language  $L$  is called **primitive positive**, or **pp-formula**, if it is of the form  $(\exists \bar{z}) \bigwedge_{i < n} \chi_i$ , where  $\chi_i$  are atomic formulas.

## Ordered version

**Question:** What can be said about ordered modules?

Ordered  $\mathbb{Z}$ -modules =  $\mathbb{Z}$ -groups = models of Presburger arithmetic (Pr)

$$(\text{Pr} = \text{Th}(\langle \mathbb{Z}, 0, 1, -, +, \leq \rangle))$$

### Theorem (Presburger)

*Every formula is in Pr equivalent to a disjunction of primitive positive formulas, i.e. to a formula of the form  $\bigvee_{i < n} (\exists \bar{z}) \psi_i$ , where each  $\psi_i$  is a system of linear inequalities.*

So the following structures have the elimination to (boolean combinations of) pp-formulas:

$$\langle M, 0, -, +, r \rangle_{r \in R} \quad \text{and} \quad \langle M, 0, 1, -, +, \leq \rangle = \langle M, 0, 1, -, +, \leq, z \rangle_{z \in \mathbb{Z}}$$

(unordered) modules                      discretely ordered  $\mathbb{Z}$ -modules

**What about ordered modules  $\langle M, 0, 1, -, +, \leq, r \rangle_{r \in R}$  with  $R \neq \mathbb{Z}$ ?**

Our main interest is in certain discretely ordered modules which arise naturally in the context of arithmetic.

## Section 2

### Linear arithmetics

# Linear arithmetic

Recall that:

**Presburger arithmetic** is the full-induction arithmetic for the language  $\langle 0, 1, -, +, \leq \rangle$ .

**Peano arithmetic** is the full-induction arithmetic for the language  $\langle 0, 1, -, +, \cdot, \leq \rangle$ .

We introduce:

**Linear arithmetic (LA)** is the full-induction arithmetic for the language  $\langle 0, 1, -, +, a \cdot, \leq \rangle$ , where  $a \cdot$  is a unary function of multiplication by a positive non-standard element.



## Higher order linear arithmetics

Similarly, for any cardinal  $\kappa$ , we introduce:

$\kappa$ -linear arithmetic ( $LA_\kappa$ ) is the full-induction arithmetic for the language  $\langle 0, 1, -, +, a_\alpha \cdot, \leq \rangle_{\alpha \in \kappa}$ , where all “scalars”  $a_\alpha$  are non-standard.

Then:

- $Pr = LA_0$ ,
- $LA = LA_1$ .

Let  $\mathcal{M} = \langle M, 0, 1, -, +, a \cdot, \leq \rangle \models LA$ .

Then  $\mathcal{M}$  can be equipped with a structure of a (discretely ordered) module over the ring  $R_a = \mathbb{Q}[a] \cap M$ .

More generally, any  $\mathcal{M} = \langle M, 0, 1, -, +, a_\alpha \cdot, \leq \rangle_{\alpha \in \kappa} \models LA_\kappa$  can be understood as a discretely ordered module over the ring  $\mathbb{Q}[a_\alpha]_{\alpha \in \kappa} \cap M$ .

**Question:** Do the pp-elimination results of Baur-Monk and Presburger generalize to arithmetics  $LA_\kappa$  with  $\kappa > 0$ ?

We show that the answer is “Yes” if and only if  $\kappa = 1$ .

The reason is that for any  $\mathcal{M} = \langle M, 0, 1, -, +, a \cdot, \leq \rangle \models LA$  (but not for models of  $LA_\kappa$  with  $\kappa \geq 2$ ), the ring  $R_a = \mathbb{Q}[a] \cap M$  is a doded.

## Section 3

### Dodeds and doded-modules

## Doded

An ordered integral domain  $D = \langle D, 0, 1, +, -, \cdot, \leq \rangle$  is called a **doded** if it

- is discretely ordered by  $\leq$ , with 1 being the least positive element,
- is regularly quasi-Euclidean, i.e. the Euclidean algorithm in  $D$  is correctly defined and always stops in finitely many steps,
- has degrees, i.e. there is a function  $\text{deg} : D \rightarrow \mathbb{N} \cup \{-\infty\}$  such that  $\text{rng}(\text{deg})$  is an initial segment of  $\mathbb{N} \cup \{-\infty\}$ , and  $\text{deg } r \leq \text{deg } q \Leftrightarrow |r| \leq n|q|$ , for some  $n \in \mathbb{N}$ .

# Doded

## Example

- 1 The ordered ring  $\mathbb{Z}$  is a doded. The degree function is given by  $\deg(z) = 0$  for  $z \neq 0$  and  $\deg(0) = -\infty$ .
- 2 For  $\mathcal{M} \models LA$ , the ring  $R_a = \mathbb{Q}[a] \cap M$  is a doded. The degree function is the degree of polynomials.

**Remark:** Rings  $R_a$  are quasi-Euclidean but not  $k$ -stage Euclidean for any  $k \in \mathbb{N}$ .

**Question:** Are there any dodeds other than those above?

# Two-sorted QE for doded-modules

Let  $\mathbb{L}$  denotes the two-sorted language of “ordered rings-ordered modules”.

A **doded-module** is a (two-sorted)  $\mathbb{L}$ -structure  $\mathcal{A} = \langle \mathcal{R}, \mathcal{M}, \cdot \rangle$  such that

- 1)  $\mathcal{R}$  is a doded,
- 2)  $\langle \mathcal{M}, r \cdot \_ \rangle_{r \in \mathcal{R}}$  is a discretely ordered (with 1 being the least positive element), integrally-divisible (i.e.  $(\forall x)(\exists y)(\exists 0 \leq z < r \cdot 1)(x = r \cdot y + z)$  holds)  $\mathcal{R}$ -module.

## Two-sorted QE for doded-modules

Let  $\mathbb{L}'$  denotes the extension of  $\mathbb{L}$  by symbols  $\bar{\cdot}^{-1} : \mathcal{R}^2 \rightarrow \mathcal{R}$ ,  
 $\bar{\cdot}^{-1} : \mathcal{R} \times \mathcal{M} \rightarrow \mathcal{M}$  for integral division.

For a doded-module  $\mathcal{A}$ , let  $\mathcal{A}'$  denotes the natural definable  $\mathbb{L}'$ -expansion of  $\mathcal{A}$ .

### Theorem

Let  $\mathcal{A} = \langle \mathcal{R}, \mathcal{M}, \cdot \rangle$  be a doded-module,  $\varphi(\bar{r}, \bar{y}, x)$  be an  $\mathbb{L}'$ -formula without scalar quantifiers, and  $\bar{\rho} \in R^{l(\bar{r})}$  be scalars. Then there are finitely many  $\mathbb{L}'$ -terms  $t_i(\bar{r}, \bar{y})$ , for  $i < n$ , with  $n \in \mathbb{N}$ , such that

$$\mathcal{A}' \models (\exists x)\varphi(\bar{\rho}, \bar{y}, x) \leftrightarrow \bigvee_{i < n} \varphi(\bar{\rho}, \bar{y}, t_i(\bar{\rho}, \bar{y})).$$



# Two-sorted QE for doded-modules

## Corollary

*Let  $\mathcal{A}$  be a doded-module. Every scalar-quantifier-free  $\mathbb{L}'$ -formula is in  $\mathcal{A}'$  equivalent to a quantifier-free  $\mathbb{L}'$ -formula.*

Let further on  $D$  be a fixed doded and  $\mathcal{M} = \langle M, 0, 1, -, +, \leq, r \rangle_{r \in D}$  be a fixed discretely ordered (with 1 being the least positive element), integrally-divisible  $D$ -module. Denote  $\mathcal{M}'$  the expansion of  $\mathcal{M}$  by definitions of functions  $r^{-1}$  providing integral division by all scalars  $0 < r \in D$ .

### Corollary

*In  $\mathcal{M}$ , every formula is equivalent to a disjunction of primitive positive formulas, i.e. to a formula of the form  $\bigvee_{i < n} (\exists \bar{z}) \psi_i$ , where each  $\psi_i$  is a system of linear inequalities.*

*Moreover, for any formula  $\varphi(x, \bar{y})$  of  $\mathcal{M}'$  there are finitely many terms (of  $\mathcal{M}'$ )  $t_i(\bar{y})$ , for  $i < n$ , with  $n \in \mathbb{N}$ , such that*

$$\mathcal{M}' \models (\exists x) \varphi(x, \bar{y}) \leftrightarrow \bigvee_{i < n} \varphi(t_i(\bar{y}), \bar{y}).$$

*Hence  $\mathcal{M}'$  has quantifier elimination.*

## A more detailed analysis

Further on, by a term or formula, we mean always an  $\mathcal{M}'$ -term or an  $\mathcal{M}'$ -formula.

We will denote by  $\mathbb{C}$  the set of all realizations of constants terms in  $\mathcal{M}'$ .

A term  $t(\bar{x})$  is **harmonic** if

$$t(\bar{x}) = \sum_{i=0}^{N-1} q_i r_i^{-1} (x_{f(i)}) + c,$$

for some  $q_i, r_i \in \mathbb{D}$ ,  $c \in \mathbb{C}$  and  $f : N \rightarrow I(\bar{x})$ .

A formula is harmonic if all its maximal subterms are.

A “piecewise-term”  $\tau$  is called an **almost-term** if it is of the form

$$\tau(\bar{x}) = \begin{cases} s(\bar{x}) + c_i & \text{if } \psi_i(\bar{x}), i < n, \end{cases}$$

where  $s(\bar{x})$  is a term, and  $c_i \in \mathbb{C}$ , for  $i < n$ .

## Harmonic form theorem

### Theorem (Harmonic form theorem)

- 1) *For every term  $t(\bar{x})$ , there is an open harmonic almost-term  $\tau(\bar{x})$  such that  $\mathcal{M}' \models t(\bar{x}) = \tau(\bar{x})$ .*
- 2) *For every formula  $\varphi(\bar{x})$ , there is an open harmonic formula  $\psi(\bar{x})$  such that  $\mathcal{M}' \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$ .*

## Representation of definable sets

### Corollary

Every set  $A \subseteq M^n$   $X$ -definable in  $\mathcal{M}'$  (for  $X \subseteq M$ ) can be written as

$$A = \bigcup_{i < k} g[P_i],$$

where  $g : (K(\bar{a}) \times M)^n \rightarrow M^n$  is a “linear coordination” of  $M^n$ ,  $\bar{a} = (a_0, \dots, a_l) \in M^l$ ,  $l \in \mathbb{N}$ , and  $P_i$ , for  $i < k$ , are finitely many polyhedra in  $(K(\bar{a}) \times M)^n$  over parameters from  $X$ .

“Every definable set is a finite union of linear images of polyhedra.”

## Section 4

### Properties of LA

The elimination results for doded-modules imply:

### Theorem (Properties of LA)

1) LA is model-complete.

Moreover: Every formula is in LA equivalent to a disjunction of primitive positive formulas, i.e. to a formula of the form  $\bigvee_{i < n} (\exists \bar{z}) \psi_i$ , where each  $\psi_i$  is a system of linear inequalities.

2)  $LA_\tau$ , for  $\tau \in \prod_{p \in \mathbb{P}} \mathbb{J}_p$ , are all simple complete extensions of LA.

For  $\mathcal{A}, \mathcal{B} \models LA$ , it is  $\mathcal{A} \equiv \mathcal{B} \Leftrightarrow a^{\mathcal{A}} \equiv a^{\mathcal{B}} \pmod{n}$ , for all  $0 < n \in \mathbb{N}$ .

3)  $R_\tau$  is the unique prime model of  $LA_\tau$ , for  $\tau \in \prod_{p \in \mathbb{P}} \mathbb{J}_p$ .

4) LA is decidable.

$LA_\tau$  is decidable if and only if  $\tau$  is recursive.

5) The induction scheme in LA may be equivalently replaced by the scheme of integral divisibility

$(\exists y, z)(x = qy + z \ \& \ z < q)$ , for all  $0 < q \in \mathbb{Z}[a]$ .

## Corollary

*Up to elementary equivalence, models of LA are exactly all ultraproducts*

$$\mathcal{Z}_{\mathcal{U}} = \left( \prod_{n \in \mathbb{N}} \langle \mathbb{Z}, 0, 1, +, -, \underline{n}, \leq \rangle \right) / \mathcal{U},$$

*where  $\mathcal{U}$  is a non-principal ultrafilter on  $\mathbb{N}$ , i.e.  $\mathcal{U} \in \beta\mathbb{N} - \mathbb{N}$ .*

**Note:** Properties of LA and Pr are similar, but the proof for LA is incomparably harder (due to [in]decomposability of sets  $\{qx + ry; 0 \leq x, y \in A\}$  in  $\mathcal{A} \models \text{Pr}[\text{LA}])$ .



The following corollary is on the structure of models of Peano arithmetic:

### Corollary

Let  $\mathcal{M} = \langle M, 0, 1, -, +, \cdot, \leq \rangle$  be a saturated model of Peano arithmetic,  $0 \leq a \in M - \mathbb{N}$ ,  $c, d \in M$ . Then the following are equivalent:

- 1) For every "Peano product"  $\circ$  on  $\mathcal{M}$  with  $a \cdot x = a \circ x$  for all  $x$ , it is  $c \cdot d = c \circ d$ .
- 2)  $c \in \mathbb{Q}[a]$  or  $d \in \mathbb{Q}[a]$ .

## Section 5

### Properties of $LA_2$ and above

## Wild models of $LA_2$

The pp-elimination, though true for  $\text{Pr} = LA_0$  and  $LA = LA_1$ , does not hold for  $LA_\kappa$  with  $\kappa \geq 2$ .

This is an easy consequence of the following:

### Proposition (with P. Pudlák)

*There is a model  $\mathcal{M} = \langle M, 0, 1, -, +, a \cdot, b \cdot, \leq \rangle \models LA_2$  and a non-standard  $l \in M$  such that  $\cdot \upharpoonright [0, l]^2$  is definable in  $\mathcal{M}$  for some Peano product  $\cdot$  on  $\mathcal{M}$  (i.e. such that  $\langle M, 0, 1, +, \cdot, \leq \rangle \models P$ ).*

Note that no such model can exist for  $LA_1$  (easy consequence of model-completeness).

### Corollary

*For  $\kappa \geq 2$ , the theory  $LA_\kappa$  does not have pp-elimination. It is not even model complete (i.e. does not have elimination to  $\exists$ -formulas).*

## Proof idea:

In a saturated model of Peano arithmetic, for any  $\mathbb{N} < I \in M$ , we find elements  $a, b$  such that the sequence  $(1, 1^2, 2, 2^2, \dots, 2I, (2I)^2)$  is encoded by the set of all nominators of convergents of the continued fraction of  $a/b$ . The crucial observation is that this set is definable in the language of  $LA_2$  with  $a, b$  as the two scalars.

(Note that for  $LA_1$  every continued fraction of scalars is finite, thus the construction above does not allow to define an infinite part of a Peano product.)

The following is well known:

### Proposition

*(Unordered) modules are stable and thus NIP.  
Presburger arithmetic (although unstable) is NIP.*

**Question (Chernikov and Hils):** Is every ordered module NIP?

### Corollary (with P. Pudlák)

*The model  $\mathcal{M}$  from the previous proposition, considered as an ordered module over  $\mathbb{Z}[a, b] \cap M$ , is not NIP.*

Despite not having the pp-elimination, the hierarchy for higher order linear arithmetics still collapses. This is a consequence of the following more general result:

Let  $\mathcal{M} = \langle 0, 1, +, -, \leq, r, \rangle$   $r \in R$  be an ordered  $R$ -module with a unit  $1 > 0$ . We say that  $\mathcal{M}$  is **i-divisible** if for every  $0 < r \in R$  it is  $\mathcal{M} \models (\forall x)(\exists y, z)(x = ry + z \ \& \ 0 \leq z < r1)$ .

### Theorem

*Let  $\mathcal{M}$  be an i-divisible ordered module. Then every formula is in  $\mathcal{M}$  equivalent to a bounded formula.*

### Corollary

*For any  $\kappa$ , every  $\mathcal{M} \models \text{LA}_\kappa$  has bounded quantifier elimination. Hence, in no model of  $\text{LA}_\kappa$  a Peano product is definable on the whole universe.*

## Proof idea:

The crucial part is to prove that in  $\mathcal{M}$ , every formula  $\psi(\bar{x})$  is **protoproperiodic**, i.e. there exists a disjoint covering  $\mathcal{A}$  of  $M^{l(\bar{x})}$  by finitely many convex polyhedra, such that for any  $A \in \mathcal{A}$  and a direction  $s = \bar{\alpha} \in R^{l(\bar{x})}$  either  $A$  is bounded in the direction  $s$  ( $\exists 0 \leq m \in M$  s.t.  $A + sm \not\subseteq A$ ) or there is a period  $0 < P \in R$  for  $\psi$  on  $A$  in the direction  $s$  (i.e.  $(\forall \bar{u}, \bar{v} \in A)((\exists m \in M)\bar{u} = Psm + \bar{v} \rightarrow (\psi(\bar{u}) \leftrightarrow \psi(\bar{v})))$ ).

This is done by induction on complexity of  $\psi$ .

## Open questions and Thank you

### Open questions:

Is there a non-trivial (i.e. not equi-definable with a model of  $LA_1$ ) model of  $LA_2$  with pp-elimination, or at least a model complete one?

Can the bounded quantifier elimination for models of  $LA_\kappa$  with  $\kappa \geq 2$  be strengthened? If yes, how does it depend on  $\kappa$ ?

Thank you for your attention.