Non-standard Methods – Three Worlds

Petr Glivický

Department of Theoretical Computer Science and Mathematical Logic
Faculty of Mathematics and Physics
Charles University in Prague

petrglivicky@gmail.com

Spring School of Algebra
Sázava, March 2012

Content

1. What is Non-standard?
2. Three Universes
3. Real Numbers and Basic Analysis

FAQ about Non-standard Methods

What is it?

– New way of looking at mathematical objects.

Does it restrict classical mathematics?

– Non-standard methods do not restrict classical mathematics in any way (in contrast with mathematical philosophies like constructivism or intuitionism). They are consistent with Zermelo-Fraenkel set theory.

What is it good for?

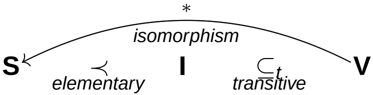
– In the non-standard world imaginary companions of an object (limit of sequence, completion of space, . . .) are already present in the object itself. This holds also for imaginaries (such as infinitely small numbers) that defy to be correctly constructed by classical methods.
– Mathematical analysis can be reformulated using infinitesimals (correctly) instead of the (ε, δ)-machinery. This makes definitions in non-standard analysis simpler, proofs more clear and also reveals truths unnoticed by classical analysis.

Three Universes

Classically all mathematical objects (considered as sets) are elements of the universal class **V** of all sets. The scheme of the universe is very simple:

$$\mathbf{V}$$

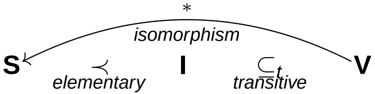
In the non-standard perspective the picture becomes little bit more complicated:



where **I** is *almost universal* and $\textcircled{\kappa}$-*saturated* for some uncountable cardinal $\textcircled{\kappa}$.

Here $\mathbf{S} \subseteq \mathbf{I} \subseteq \mathbf{V}$ are three classes called the standard, internal and external universe respectively.

Three Universes



I is $\textcircled{\kappa}$-saturated and almost universal

By $\in$-formula we further mean a formula in the language of set theory. In particular classes **S**, **I** nor the mapping * do not appear in $\in$-formulas. For φ an $\in$-formula $\varphi^{\mathbf{X}}$ is the relativization of φ to **X** obtained by replacing all quantifiers $\forall x, \exists x$ in φ by $\forall x \in \mathbf{X}, \exists x \in \mathbf{X}$.

Definition

$^* : \mathbf{V} \to \mathbf{S}$ isomorphism	$\Leftrightarrow$	$\varphi^{\mathbf{V}}(\bar{a}) \Leftrightarrow \varphi^{\mathbf{S}}(^*\bar{a})$ for $\in$-formula φ, $\bar{a} \in \mathbf{V}$, * is injective and onto $\mathbf{S} = \mathrm{rng}(^*) = {}^*[\mathbf{V}]$,
$\mathbf{S} \prec \mathbf{I}$ (elementary)	$\Leftrightarrow$	$\varphi^{\mathbf{S}}(\bar{a}) \Leftrightarrow \varphi^{\mathbf{I}}(\bar{a})$ for $\in$-formula φ and $\bar{a} \in \mathbf{S}$,
$\mathbf{I} \subseteq_t \mathbf{V}$ (transitive)	$\Leftrightarrow$	$y \in x \in \mathbf{I} \to y \in \mathbf{I}$,
I almost universal	$\Leftrightarrow$	$x \subseteq \mathbf{I} \to (\exists y \in \mathbf{I})(x \subseteq y)$,
I $\textcircled{\kappa}$-saturated	$\Leftrightarrow$	$(\mathcal{C} \subseteq \mathbf{I}$ centered & $\lvert\mathcal{C}\rvert < \textcircled{\kappa}) \to \bigcap \mathcal{C} \neq \emptyset$ where a system $\mathcal{C}$ is centered if for any finite $\mathcal{C}' \subseteq \mathcal{C}$ it is $\bigcap \mathcal{C}' \neq \emptyset$

Three Universes

$\mathbf{S} \underset{\textit{elementary}}{\prec} \mathbf{I} \underset{\textit{transitive}}{\subseteq_t} \mathbf{V}$, with $\mathbf{V} \xrightarrow[\textit{isomorphism}]{*} \mathbf{S}$ — **I** is $\textcircled{\kappa}$-saturated and almost universal

---

The standard universe **S** is a copy of **V**. The internal universe **I** is extension of **S** which contains imaginary objects for elements of **S**.

For a set x we say that x is standard resp. internal if $x \in \mathbf{S}$ resp. $x \in \mathbf{I}$. For instance we may say that r is standard real number or that f is internal function.

Similarly we say that an $\in$-formula $\varphi(\overline{a})$ holds standardly resp. internally if $\overline{a}$ is standard resp. internal and $\varphi^{\mathbf{S}}(\overline{a})$ resp. $\varphi^{\mathbf{I}}(\overline{a})$ holds. We may then say e.g. that a is an internally finite set (and write it as $[a \text{ is finite}]^{\mathbf{I}}$).

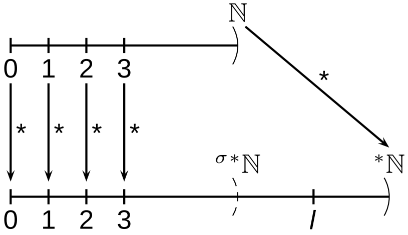
Standard extension of a set x is the set ${}^{\sigma}x = x \cap \mathbf{S}$ of standard elements of x. Standardization ${}^{\mathrm{ST}}x$ of $x \subseteq \mathbf{S}$ is the unique standard set y with ${}^{\sigma}y = x$.

Natural numbers in the non-standard perspective

For the set $\mathbb{N} \in \mathbf{V}$ of natural numbers there is its "standard instance" ${}^*\mathbb{N} \in \mathbf{S}$ and the pointwise copy ${}^*[\mathbb{N}] = {}^{\sigma *}\mathbb{N}$. It is ${}^*n = n$ for $n \in \mathbb{N}$ and therefore ${}^{\sigma *}\mathbb{N} = \mathbb{N}$.

Since * is isomorphism of **S** and **V**, ${}^*\mathbb{N}$ is standardly the set of all natural numbers. For $\mathbf{S} \prec \mathbf{I}$ and ${}^*\mathbb{N} \in \mathbf{S}$, ${}^*\mathbb{N}$ is also internally the set of all natural numbers.

$\mathcal{C} = \{{}^*\mathbb{N} - \{n\}; n \in \mathbb{N}\}$ is a centered system of internal sets of size $\omega < \textcircled{\kappa}$. By $\textcircled{\kappa}$-saturation of **I** there is $I \in \bigcap \mathcal{C}$, clearly $I \in {}^*\mathbb{N} - \mathbb{N}$.



${}^*n = n \in \mathbf{S}$ for $n \in \mathbb{N}$

$I \in \mathbf{I} - \mathbf{S}$ infinite natural number

${}^*\mathbb{N} \in \mathbf{S}$ standard set of nat. numbers

${}^{\sigma *}\mathbb{N} = \mathbb{N} \notin \mathbf{I}$ set of standard natural numbers

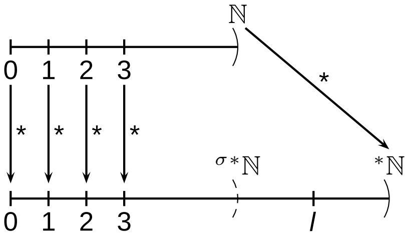
${}^*\mathbb{N} - \mathbb{N} \notin \mathbf{I}$ set of non-standard natural numbers

Universal ⓚ-saturated elementary extension

For the structure $\underline{\mathbb{N}} = \langle \mathbb{N}, 0, S, +, \cdot, \leq \rangle$ of natural numbers it is ${}^*\underline{\mathbb{N}} = \langle {}^*\mathbb{N}, {}^*0, {}^*S, {}^*+, {}^*\cdot, {}^*\leq \rangle$ and ${}^*\underline{\mathbb{N}}$ is ⓚ-saturated elementary extension of $\underline{\mathbb{N}}$.

Arbitrary structure $\mathcal{A}$ in a finite language can be elementarily embedded into ${}^*\mathcal{A}$ in the similar way and ${}^*\mathcal{A}$ is ⓚ-saturated structure. Hence * gives an universal way for construction of ⓚ-saturated elementary extensions of structures.

Natural numbers in the non-standard perspective



$^*n = n \in$ **S** for $n \in \mathbb{N}$

$I \in$ **I** $-$ **S** infinite natural number

$^*\mathbb{N} \in$ **S** standard set of nat. numbers

$^{\sigma*}\mathbb{N} = \mathbb{N} \notin$ **I** set of standard natural numbers

$^*\mathbb{N} - \mathbb{N} \notin$ **I** set of non-standard natural numbers

The set $^*\mathbb{N} - \mathbb{N}$ is a subset of $^*\mathbb{N}$ without the least element. This is no contradiction at all.

$^*\mathbb{N}$ is not “the set of all natural numbers” externally. It is “the set of all natural numbers” standardly and internally but $^*\mathbb{N} - \mathbb{N} \notin$ **I** (actually this is the proof of that). Hence we have also $\mathbb{N} \notin$ **I**.

Non-standard Principles

The fact that $\mathbb{N} \subseteq {}^*\mathbb{N}$ but $\mathbb{N}$ is not internal can be reformulated as follows:

Proposition (Overflow Principle)

1. $\mathbb{N} \subseteq x \in \mathbf{I} \to x \cap ({}^*\mathbb{N} - \mathbb{N}) \neq \emptyset$
2. *For $\bar{a} \in \mathbf{I}$ and φ an $\in$-formula it is*

$$(\forall x \in \mathbb{N})\varphi^{\mathbf{I}}(x, \bar{a}) \to (\exists x \in {}^*\mathbb{N} - \mathbb{N})\varphi^{\mathbf{I}}(x, \bar{a}).$$

The concept of finiteness is not absolute for the nonstandard universes. For example any $I \in {}^*\mathbb{N} - \mathbb{N}$ is internally finite ($[|I| = I]^{\mathbf{I}}$) but infinite (even $|I| \geq ⓚ$). The following holds:

Proposition (ⓚ-finitarization Principle)

$$(x \subseteq \mathbf{I} \;\&\; |x| < ⓚ) \to (\exists y \in \mathbf{I})([y \text{ is finite}]^{\mathbf{I}} \;\&\; x \subseteq y)$$

Application of the Finitarization Principle

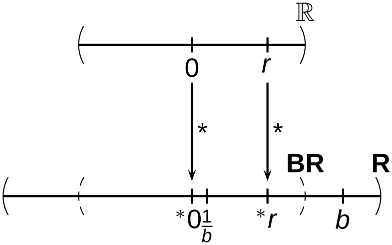
Let $G = \langle V, E \rangle$ be a graph such that each of its finite subgraphs is colorable by at most k-colors (k-colorable). We prove that G is k-colorable.

Proof: Suppose $|V| < \text{ⓚ}$. Take ${}^*G = \langle {}^*V, {}^*E \rangle$. By identifying $v \in V$ with ${}^*v \in {}^*V$ we may assume that G is a subgraph of ${}^*G$.

By ⓚ-finitarization principle there is an internally finite set Y with $V \subseteq Y \subseteq {}^*V$. Then internally $\langle Y, {}^*E \restriction Y \rangle$ is a finite subgraph of ${}^*G$ and hence k-colorable. The restriction of the coloring to $V \subseteq Y$ is the k-coloring of G.

Real Numbers

Denote $\mathbf{R} = {}^*\mathbb{R}$ the standard set of real numbers. We will identify $\mathbb{R}$ with the subset ${}^\sigma\mathbf{R} = \{{}^*r; r \in \mathbb{R}\} \subseteq \mathbf{R}$ of standard real numbers. Then we have the following:



b is infinitely big

$1/b$ is infinitely small

$1/b \in \mathbf{BR} - {}^\sigma\mathbf{R}$

$\mathbf{BR} = \{r \in \mathbf{R}; |r| < s \text{ for some } s \in {}^\sigma\mathbf{R}\}$ is the set of bounded real numbers.

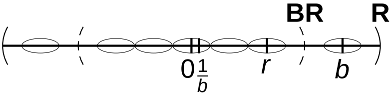
Definition

The indifference $\sim$ on **R** is defined as follows:

$$x \sim y \Leftrightarrow |x - y| < 2^{-n} \text{ for all } n \in \mathbb{N}$$

If $x \sim y$, we say that x and y are *infinitely close* to each other.

The indifference $\sim$ is an equivalence and hence gives a partition of **R**. The equivalence classes $[x]_\sim$ are called monads.



it is $1/b \sim 0$

Theorem (Fundamental about **R**)

For every $r \in$ **BR** *there is standard* $s \in \mathbb{R}$ *such that* $r \sim s$.

Non-standard Analysis

Definition (continuous and uniformly continuous function)

We say that a standard (partial) function $f \subseteq \mathbf{R} \times \mathbf{R}$ is continuous in a standard point $a \in \mathrm{dom}(f)$ if for every $x \in \mathrm{dom}(f)$ it is

$$a \sim x \to f(a) \sim f(x). \tag{1}$$

Standard function $f \subseteq \mathbf{R} \times \mathbf{R}$ is continuous if (1) holds for every standard $a \in \mathrm{dom}(f)$ and it is uniformly continuous if (1) holds for every $a \in \mathrm{dom}(f)$.

Definition (derivative)

For a standard function $f \subseteq \mathbf{R} \times \mathbf{R}$ and a such that $[a]_\sim \subseteq \mathrm{dom}(f)$ we say that $b \in \mathbb{R}$ is the derivative of f in a if

$$0 \neq \delta \sim 0 \to \frac{f(a+\delta) - f(a)}{\delta} \sim b.$$

Strange functions

Let $sin : \mathbf{R} \to \mathbf{R}$ be the standard sinus function and $l \in {}^*\mathbb{N} - \mathbb{N}$. Define $\sigma : \mathbb{R} \to \mathbb{R}$ as the "standard trace" of $sin(lx)$, i.e. for $r \in \mathbb{R}$ let $\sigma(r)$ be the unique standard element of the monad $[sin(lr)]_{\sim}$.

Proposition

The function σ is not measurable.

Another "strange" function can be defined as follows.

Suppose $\textcircled{\kappa} > |\mathbb{R}|$. Then by the $\textcircled{\kappa}$-finitarization principle there is $y \in \mathbf{I}$ internally finite such that $\mathbb{R} \subseteq y \subseteq \mathbf{R}$. Let internally $(y)_i$, $i < |y|$ be the ascending enumeration of y. Let $f : \mathbb{R} \mapsto \{0, 1\}$ is defined as

$$f(y_i) = \begin{cases} 0 & \textit{for } i \textit{ even} \\ 1 & \textit{for } i \textit{ odd} \end{cases}.$$

Differential equations

Let standardly $u'(t) = f(u(t))$ be an ordinary differential equation with the condition $u(t_0) = a$ and with f bounded. We show the idea of finding the solution on an interval $[t_0, t_0 + r)$ using non-standard methods.

Idea: First find an internal "infinitesimal difference solution". Let $l \in {}^*\mathbb{N} - \mathbb{N}$ and $\Delta = r/l$; it is $\Delta \sim 0$. Now internally: For $i < l$ let $t_i = t_0 + i \cdot \Delta$ and (y_i), $i < l$ is defined recursively $y_0 = a$, $y_i = y_{i-1} + \Delta \cdot f(y_{i-1})$.

Then $U : t_i \mapsto y_i$ is a partial function. Since f is standardly bounded by some $M \in$ **BR**, for $t_i \sim t_j$, $i < j$ we have $|y_j - y_i| = |\sum_{k=i}^{j-1} \Delta \cdot f(y_k)| \leq (j - i) \cdot \Delta \cdot M = (t_i - t_j) \cdot M \sim 0$.

We define the "standard trace" of U. For $x \in [t_0, t_0 + r)$ standard let i be an index such that $t_i \sim x$ and let $y(x)$ be the unique standard element of the monad $[y_i]_\sim$. Then $u = {}^*y$ is the solution of the original differential equation.

Thanks

Thank you for your attention.