Nonstandard methods in combinatorics on $\mathbb{N}$

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Content

Iterated *

Recall that for $n \in \mathbb{N}$ the mapping ${}^{n*} : \mathbf{V} \to \mathbf{V}$ is * applied n-times.

So, in particular, ^{0*} is the identity and ^{1*} is just *.

(**Note:** This definition is by induction in set theory, i. e. we define ^{n*} not just for metamathematical n but for all $n \in \omega$.)

Furthermore, we define ${}^{\bullet} : \mathbf{V} \to \mathbf{V}$ by

$$ {}^{\bullet}x = \bigcup_{n \in \mathbb{N}} {}^{n*}x. $$

Nonstandard extensions of $\mathbb{N}$

We have the following chain of nonstandard extensions of $\mathbb{N}$:

$$\mathbb{N} \subsetneq {}^{*}\mathbb{N} \subsetneq {}^{**}\mathbb{N} \subsetneq \cdots \subsetneq {}^{\bullet}\mathbb{N}.$$

The sets ${}^{n*}\mathbb{N}$ are initial segments of ${}^{\bullet}\mathbb{N}$ with respect to the usual ordering $\in$ of natural numbers.

See the board for the picture.

Letters α, β, ν denote elements of ${}^{\bullet}\mathbb{N}$, $\overline{\alpha}, \overline{\beta}, \overline{\nu}$ tuples of elements from ${}^{\bullet}\mathbb{N}$.

The rank function on ${}^{\bullet}\mathbb{N}$ is defined as

$$r(\alpha) = \min\{n \in \mathbb{N}; \alpha \in {}^{n*}\mathbb{N}\}, r(\overline{\alpha}) = \max\{r(\alpha_i); i < l(\overline{\alpha})\}.$$

Nonstandard extensions of $\mathbb{N}$

Lemma (Transfer principle for ${}^{\bullet}\mathbb{N}$)

Let φ be bounded $\in$-formula, $\overline{\alpha} \in {}^{\bullet}\mathbb{N}$, $m, n \geq r(\overline{\alpha})$. Then

$$\varphi(\overline{\alpha}, {}^{m*}\overline{y}) \leftrightarrow \varphi(\overline{\alpha}, {}^{n*}\overline{y}).$$

Proof: By $n - m$ iterations of the transfer principle, we get

$$(\varphi(\bar{i}, \overline{y}) \leftrightarrow \varphi(\bar{i}, {}^{(n-m)*}\overline{y}))$$

for all $\bar{i} \in \mathbb{N}$.

By applying the transfer principle m times on the bounded formula

$$(\forall \bar{i} \in \mathbb{N})(\varphi(\bar{i}, \overline{y}) \leftrightarrow \varphi(\bar{i}, {}^{(n-m)*}\overline{y}))$$

we get

$$(\forall \overline{\alpha} \in {}^{m*}\mathbb{N})(\varphi(\overline{\alpha}, {}^{m*}\overline{y}) \leftrightarrow \varphi(\overline{\alpha}, {}^{n*}\overline{y})),$$

and we are done because $\overline{\alpha} \in {}^{r(\overline{\alpha})*}\mathbb{N} \subseteq {}^{m*}\mathbb{N}$.

Corollary

Let $\mathcal{S}$ denote the set of all functions $g : \mathbb{N}^m \to \mathbb{N}$ and relations $R \subseteq \mathbb{N}^m$ of all arities m. The structures $\langle {}^{n*}\mathbb{N}, {}^{n*}s \rangle_{s\in\mathcal{S}}$ form the elementary chain

$$\langle \mathbb{N}, s \rangle_{s\in\mathcal{S}} \prec \langle {}^{*}\mathbb{N}, {}^{*}s \rangle_{s\in\mathcal{S}} \prec \langle {}^{**}\mathbb{N}, {}^{**}s \rangle_{s\in\mathcal{S}} \ldots \prec \langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s \rangle_{s\in\mathcal{S}}$$

with the limit $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s \rangle_{s\in\mathcal{S}}$.

Proof: Observe that for any $\mathcal{S}$-formula ψ, the formula

$$\langle {}^{n*}\mathbb{N}, {}^{n*}s \rangle_{s\in\mathcal{S}} \models \psi(\overline{\alpha})$$

is equivalent to a bounded $\in$-formula

$$\varphi(\overline{\alpha}, {}^{n*}\mathbb{N}, {}^{n*}\overline{s})$$

where $\overline{s} \in S$ is a finite tuple containing all symbols from S occuring in ψ.

Note: Above, ψ is a mathematical formula (i.e. a formula in the sense of set theory), while φ is a metamathematical $\in$-formula (with just three variables).

Indiscernibles

An increasing sequence $(\beta_i; i \in \omega)$ of elements from ${}^{\bullet}\mathbb{N}$ is indiscernible with respect to the formula $\varphi(\overline{x})$ if

$$\varphi(\beta_{i_0}, \ldots, \beta_{i_{l(\overline{x})-1}}) \leftrightarrow \varphi(\beta_{j_0}, \ldots, \beta_{j_{l(\overline{x})-1}})$$

whenever $i_0 < \ldots < i_{l(\overline{x})-1}$, $j_0 < \ldots < j_{l(\overline{x})-1}$.

We also say that $(\beta_i; i \in \omega)$ is strongly indiscernible with respect to the formula $\varphi(\overline{z}, \overline{x})$ if

$$(\forall \overline{z} \leq \beta_k)(\varphi(\overline{z}, \beta_{i_0}, \ldots, \beta_{i_{l(\overline{x})-1}}) \leftrightarrow \varphi(\overline{z}, \beta_{j_0}, \ldots, \beta_{j_{l(\overline{x})-1}}))$$

whenever $k < i_0 < \ldots < i_{l(\overline{x})-1}$, $k < j_0 < \ldots < j_{l(\overline{x})-1}$.

The sequence $(\beta, {}^{r(\beta)*}\beta, {}^{2r(\beta)*}\beta, \ldots)$ with $\beta \in {}^\bullet\mathbb{N}$ satisfies an indiscernibility property:

Proposition

Let $\beta \in {}^\bullet\mathbb{N}$ and φ be a bounded $\in$-formula. Then

$$\varphi(\overline{\alpha}, {}^{i_0 r(\beta)*}\beta, \ldots, {}^{i_{k-1} r(\beta)*}\beta, {}^{m*}\overline{y}) \leftrightarrow \varphi(\overline{\alpha}, {}^{i'_0 r(\beta)*}\beta, \ldots, {}^{i'_{k-1} r(\beta)*}\beta, {}^{m'*}\overline{y})$$

for all $i_0 < \cdots < i_{k-1}$, $i'_0 < \cdots < i'_{k-1}$, all $\overline{\alpha}$ satisfying $r(\overline{\alpha}) \leq i_0 r(\beta), i'_0 r(\beta)$ and all m, m' such that $(i_{k-1}+1)r(\beta) \leq m$, $(i'_{k-1}+1)r(\beta) \leq m'$.

Proof: Technical, but just by application of the transfer principle for ${}^\bullet\mathbb{N}$.

We say that $(\beta_i; i \in \omega)$ is rank-indiscernible with respect to the formula $\varphi(\overline{z}, \overline{x})$ if

$$(\forall \overline{z})(r(\overline{z}) \leq r(\beta_k) \rightarrow (\varphi(\overline{z}, \beta_{i_0}, \ldots, \beta_{i_{l(\overline{x})-1}}) \leftrightarrow \varphi(\overline{z}, \beta_{j_0}, \ldots, \beta_{j_{l(\overline{x})-1}})))$$

whenever $k < i_0 < \ldots < i_{l(\overline{x})-1}$, $k < j_0 < \ldots < j_{l(\overline{x})-1}$.

Clearly any rank-indiscernible sequence is strongly indiscernible.

We say that a sequence is [strogly, rank-] indiscernible with respect to a formula $\varphi(\overline{x})$ of the structure $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s\rangle_{s\in\mathcal{S}}$ (i.e. $\varphi(\overline{x})$ is an S-formula) if it is [strongly, rank-] indiscernible with respect to the $\in$-formula $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s\rangle_{s\in\mathcal{S}} \models \varphi(\overline{x})$.

Corollary

For every $\beta \in {}^{\bullet}\mathbb{N}$, the sequence $(\beta, {}^{r(\beta)*}\beta, {}^{2r(\beta)*}\beta, \ldots)$ is rank-indiscernible with respect to all

1) *parameter-free bounded $\in$-formulas,*
2) *parameter-free formulas of $\langle {}^{\bullet}\mathbb{N}, {}^{\bullet}s\rangle_{s\in\mathcal{S}}$.*

Extensions of functions and relations

We have already defined ${}^{\bullet}g$ and ${}^{\bullet}R$ for all (partial) functions $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$ and relations $R \subseteq \mathbb{N}^{<\omega}$.
(Note that ${}^{\bullet}g$ and ${}^{\bullet}R$ may extend beyond ${}^{\bullet}\mathbb{N}^{<\omega}$. Nevertheless, we are only interested in their restrictions to ${}^{\bullet}\mathbb{N}^{<\omega}$.)

For the sake of brevity, we will frequently write just $g(\overline{\alpha})$ instead of ${}^{\bullet}g(\overline{\alpha})$ and $R(\overline{\alpha})$ instead of ${}^{\bullet}R(\overline{\alpha})$

By the transfer principle on ${}^{\bullet}\mathbb{N}$ we get:

$${}^{\bullet}g(\overline{\alpha}) = {}^{r(\overline{\alpha})*}g(\overline{\alpha}), \quad {}^{\bullet}R(\overline{\alpha}) \leftrightarrow {}^{r(\overline{\alpha})*}R(\overline{\alpha}).$$

Grading

We define the transformation (grading) $^{\uparrow} : {}^{\bullet}\mathbb{N}^{<\omega} \to {}^{\bullet}\mathbb{N}^{<\omega}$ by

$$\overline{\alpha}^{\uparrow} = (\alpha_0, {}^{r(\alpha_0)*}\alpha_1, {}^{(r(\alpha_0)+r(\alpha_1))*}\alpha_2, \dots, {}^{(\sum_{i<l(\overline{\alpha})-1} r(\alpha_i))*}\alpha_{l(\overline{\alpha})-1}).$$

For every (partial) function $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$ and every relation $R \subseteq \mathbb{N}^{<\omega}$ we set

$$g^{\uparrow}(\overline{\alpha}) = {}^{\bullet}g(\overline{\alpha}^{\uparrow}), \quad R^{\uparrow}(\overline{\alpha}) \leftrightarrow {}^{\bullet}R(\overline{\alpha}^{\uparrow}),$$

for all $\overline{\alpha} \in {}^{\bullet}\mathbb{N}$ such that the right side is defined.

In particular, we get:

$$\alpha +^{\uparrow} \beta = \alpha + {}^{r(\alpha)*}\beta, \quad \alpha \cdot^{\uparrow} \beta = \alpha \cdot {}^{r(\alpha)*}\beta.$$

Indistinguishability equivalence on ${}^{\bullet}\mathbb{N}$

An important equivalence on ${}^{\bullet}\mathbb{N}$ is that of indistinguishability by standard properties:

$$\alpha \sim \beta \leftrightarrow (\forall A \subseteq \mathbb{N})(\alpha \in {}^{\bullet}A \leftrightarrow \beta \in {}^{\bullet}A).$$

Note that $\sim$ is nontrivial (for example $\alpha \sim {}^{n*}\alpha$ for all n). In fact $\sim$ corresponds to the Čech-Stone compactification $\beta\mathbb{N}$ of $\mathbb{N}$.

We say that an equivalence $\approx$ on ${}^{\bullet}\mathbb{N}$ is a congruence with respect to a (partial) function $f : {}^{\bullet}\mathbb{N}^{<\omega} \nrightarrow {}^{\bullet}\mathbb{N}$ or a relation $P \subseteq {}^{\bullet}\mathbb{N}^{<\omega}$ if

$$\overline{\alpha} \approx \overline{\alpha}' \rightarrow f(\overline{\alpha}) \approx f(\overline{\alpha}') \quad \text{or} \quad \overline{\alpha} \approx \overline{\alpha}' \rightarrow (P(\overline{\alpha}) \leftrightarrow P(\overline{\alpha}'))$$

respectively, granted that, in the first case, at least one of $f(\overline{\alpha})$, $f(\overline{\alpha}')$ is defined.

Proposition

The equivalence $\sim$ is a congruence with respect to $s^{\uparrow}$ whenever s is a partial function $s : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$ or a relation $s \subseteq \mathbb{N}^{<\omega}$.

Indistinguishability equivalence on ${}^{\bullet}\mathbb{N}$

As for unary $g : \mathbb{N} \to \mathbb{N}$ we have by definition $g^{\uparrow} = g$, we get the following trivial but important:

Corollary

Let $g : \mathbb{N} \to \mathbb{N}$ then $\sim$ is a congruence with respect to g.

For binary g however this is not true anymore. The equivalence $\sim$ is not congruence with respect to operations $+$ or $\cdot$:

Example: For $\nu \in {}^{\bullet}\mathbb{N}$ denote by $v_2(\nu)$ the largest μ such that $2^{\mu}|\nu$. Let $\alpha \in {}^{*}\mathbb{N} - \mathbb{N}$ be such that $\beta = v_2(\alpha) \notin \mathbb{N}$. Then $\alpha \sim {}^{*}\alpha$ but $\alpha + \alpha \not\sim \alpha + {}^{*}\alpha$. Let us prove the latter claim:
We have $v_2({}^{*}\alpha) = {}^{*}\beta > \beta$ and therefore $v_2(\alpha + {}^{*}\alpha) = \beta$. On the other hand $v_2(\alpha + \alpha) = v_2(2\alpha) = \beta + 1$. Define $A = \{x \in \mathbb{N}; v_2(x) \text{ is odd}\}$. Then we have $\alpha + \alpha \in {}^{*}A \leftrightarrow \alpha + {}^{*}\alpha \notin {}^{**}A$.

Indistinguishability equivalence on ${}^{\bullet}\mathbb{N}$

The following proposition illustrates, how the equivalence classes of $\sim$ are placed inside ${}^{\bullet}\mathbb{N}$:

Proposition ($\textcircled{\kappa} > 2^{\omega}$)

For every class C of the equivalence $\sim$ exactly one of the following statements holds true:

- $C = \{n\}$ *for some* $n \in \mathbb{N}$,
- $C \cap \mathbb{N} = \emptyset$ *and* $C \cap \left({}^{(n+1)*}\mathbb{N} - {}^{n*}\mathbb{N}\right) \neq \emptyset$ *for all* $n \in \mathbb{N}$.

Topology

We define

$$\tilde{\mathbb{N}} = {}^{\bullet}\mathbb{N}/_{\sim},\ \tilde{g} = g^{\uparrow}/_{\sim},\ \tilde{R} = R^{\uparrow}/_{\sim},\ \tilde{\alpha} = \alpha/_{\sim}$$

for $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$, $R \subseteq \mathbb{N}^{<\omega}$, $\alpha \in {}^{\bullet}\mathbb{N}$, and $\Upsilon : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\tilde{\mathbb{N}})$ by

$$\Upsilon(A) = \{\tilde{\alpha}; \alpha \in {}^{\bullet}A\}.$$

The set $\{\Upsilon(A); A \subseteq \mathbb{N}\}$ is a basis of a topology on $\tilde{\mathbb{N}}$. We call this topology the canonical topology on $\tilde{\mathbb{N}}$. The canonical topology is uniform with the basis of uniformity

$$\tilde{\mathbb{W}} = \{\tilde{w}_t; t \in [\mathcal{P}(\mathbb{N})]^{<\omega}\}\ ;\ \tilde{w}_t = \{(\tilde{\alpha}, \tilde{\beta}); (\forall A \in t)(\tilde{\alpha} \in \Upsilon(A) \leftrightarrow \tilde{\beta} \in \Upsilon(A))\}.$$

Proposition (ⓚ $> 2^{\omega}$)

1) $(\tilde{\mathbb{N}}, \tilde{\mathbb{W}})$ *is a uniform compact Hausdorff space.*
2) *All functions* $\tilde{g}$ *with* $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$ *are continuous in the first coordinate in* $(\tilde{\mathbb{N}}, \tilde{\mathbb{W}})$.

Idempotent numbers

We say that $\nu \in {}^{\bullet}\mathbb{N}$ is g-idempotent (with $g : \mathbb{N}^{<\omega} \nrightarrow \mathbb{N}$) if $\tilde{\nu}$ is an idempotent element of $(\tilde{\mathbb{N}}, \tilde{g})$, i.e. if $\tilde{g}(\tilde{\nu}, \ldots, \tilde{\nu}) = \tilde{\nu}$ whenever the left side is defined.

That is, ν is g-idempotent iff $g^{\uparrow}(\nu, \ldots, \nu) \sim \nu$.

In particular: A number ν is additively or multiplicatively idempotent if

$$\nu +^{\uparrow} \nu = \nu + {}^{r(\nu)*}\nu \sim \nu \text{ or } \nu \cdot^{\uparrow} \nu = \nu \cdot {}^{r(\nu)*}\nu \sim \nu$$

respectively.

Existence of idempotents

A semigroup is a structure with a single binary associative operation.

Lemma (Ellis-Numakura)

Let S be a semigroup with a compact Hausdorff topology and such that the semigroup operation is left-continuous. Then S contains an idempotent element.
Moreover, every element of the minimal compact subsemigroup of S is idempotent.

Corollary (Ⓚ $> 2^{\omega}$)

There is an additively [a multiplicatively] idempotent element $\mu \neq 0$ [$0 \neq \nu \neq 1$] in ${}^{*}\mathbb{N}$.

Proof: By Ellis'-Numakura's Lemma applied on the semigroup $\langle \tilde{\mathbb{N}} - \{\tilde{0}\}, \tilde{+} \rangle$ [$\langle \tilde{\mathbb{N}} - \{\tilde{0}, \tilde{1}\}, \tilde{\cdot} \rangle$], we get that there is an additively [a multiplicatively] idempotent element $\mu \neq 0$ [$0 \neq \nu \neq 1$] in ${}^{\bullet}\mathbb{N}$. Let $\nu \sim \mu$ be such that $\nu \in {}^{*}\mathbb{N}$. Then ν is idempotent as well.

Idempotence for function composition

For operations f, g on $\mathbb{N}$ and $n \leq ar(f)$ we define their order preserving composition in the n-th variable

$$(f \circ_n g)(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = f(\overline{\alpha}, g(\overline{\beta}), \overline{\gamma})$$

for all $\overline{\alpha} \in \mathbb{N}^{n-1}$, $\overline{\beta} \in \mathbb{N}^{ar(g)}$, $\overline{\gamma} \in \mathbb{N}^{ar(f)-n}$. (Simply said: g is substituted onto the nth position of f.)

Note that the variables appear on both sides of the defining equation in the same order. This is important as it guarantees that $\circ_n$ behaves well with regard to gradation:

Lemma

$(f \circ_n g)^{\uparrow} = f^{\uparrow} \circ_n g^{\uparrow}$.

Proof: Easy computation.

Idempotence for function composition

For a set $\mathcal{G}$ of operations on $\mathbb{N}$, the set $\overline{\mathcal{G}}^{op}$ of all order preserving compositions from $\mathcal{G}$ is defined as the smallest set containing $\mathcal{G}$ and closed under all $\circ_n$.

Corollary

If ν is a common idempotent of $\mathcal{G}$, then it is a common idempotent of $\overline{\mathcal{G}}^{op}$ as well.

Proof: By induction: Suppose ν is idempotent for f, g. Then

$$(f \circ_n g)^{\uparrow}(\overline{\nu}) = (f^{\uparrow} \circ_n g^{\uparrow})(\overline{\nu}) = f^{\uparrow}(\overline{\nu}, g^{\uparrow}(\overline{\nu}), \overline{\nu}) \sim f^{\uparrow}(\overline{\nu}, \nu, \overline{\nu}) \sim \nu,$$

where each $\overline{\nu}$ denotes a constant tupple $(\nu, \ldots \nu)$ of an appropriate length.

The fullness problem

Notation: For $E \subseteq \mathbb{N}$ and X a set we denote by $\langle X\rangle^{\in E}$ the set $\bigcup_{e\in E}\langle X\rangle^{e}$. By $\overline{x}^n$ we mean the constant tupple $(x,\dots,x)$ of length n.

One of the basic questions in Ramsey type combinatorics is the question of fullness:

Let $E \subseteq \mathbb{N}$. The set $C \subseteq \langle\mathbb{N}\rangle^{\in E}$ is called E-full (full in dimensions E) if there is $A \in [\mathbb{N}]^{\omega}$ such that $\langle A\rangle^{\in E} \subseteq C$.

In this section, we provide a general framework for describing nonstandard equivalents of E-fullness, and we give explicit (and very useful in applications) nonstandard characterizations of E-fullness for E finite and for $E = \mathbb{N} - n$ with $0 \neq n \in \mathbb{N}$.

The nonstandard equivalents of fullness

Theorem

1) *Let $E \subseteq \mathbb{N}$ be finite, $n = \max(E)$. Then the following are equivalent for every $C \subseteq \langle\mathbb{N}\rangle^{\in E}$:*
   a) $(\exists A \in [\mathbb{N}]^{\omega})\langle A\rangle^{\in E} \subseteq C$,
   b) $(\exists \nu \in {}^{\bullet}\mathbb{N} - \mathbb{N})(\forall e \in E)(\overline{\nu}^{e} \in C^{\uparrow})$.

2) *Let $E = \mathbb{N} - n$ for some $0 \neq n \in \mathbb{N}$. Then the following are equivalent for every $C \subseteq \langle\mathbb{N}\rangle^{\in E}$:*
   a) $(\exists A \in [\mathbb{N}]^{\omega})\langle A\rangle^{\in E} \subseteq C$
   b) $(\exists X \in [\mathbb{N}]^{\omega})(\exists \nu \in {}^{\bullet}X - X)(\overline{\nu}^{n} \in C^{\uparrow}\,\&\,(\forall F \in \langle X\rangle^{<\omega})(F \smile \overline{\nu}^{n-1} \in C^{\uparrow} \rightarrow F \smile \overline{\nu}^{n} \in C^{\uparrow}))$.

Note: Both of the criteria are just special cases of single more general statement that is little bit too abstract to be presented here. Therefore, we prove them separately:

The nonstandard equivalents of fullness

Theorem

1) *Let $E \subseteq \mathbb{N}$ be finite, $n = \max(E)$. Then the following are equivalent for every $C \subseteq \langle\mathbb{N}\rangle^{\in E}$:*

   a) $(\exists A \in [\mathbb{N}]^{\omega})\langle A\rangle^{\in E} \subseteq C$,
   b) $(\exists \nu \in {}^{\bullet}\mathbb{N} - \mathbb{N})(\forall e \in E)(\overline{\nu}^{e} \in C^{\uparrow})$.

Proof: a)⇒b): Any $\nu \in {}^{\bullet}A - \mathbb{N}$ has the required property as $(\overline{\nu}^{e})^{\uparrow} \in \langle {}^{\bullet}A\rangle^{\in E} \subseteq {}^{\bullet}C$.
b)⇒a): We construct $A = \{a_0, a_1, \dots\} \subseteq \mathbb{N}$ with $a_0 < a_1 < \cdots$ such that

$$(\forall e \in E)(\forall F \in \langle A\rangle^{\leq e})(F \smile \overline{\nu}^{e-|F|} \in C^{\uparrow}). \qquad (\#)$$

Then in particular $\langle A\rangle^{\in E} \in C$.

Let $A' = a_0, \dots, a_{n-1}$ satisfies $(\#)$, we find a_n:

$$(\forall e \in E)(\forall F \in \langle A' \rangle^{\leq e})(F \smile \overline{\nu}^{e-|F|} \in C^{\uparrow}). \qquad (\#)$$

For $e \in E$ and $F \in \langle A' \rangle^{<e}$ we have:

$$\begin{aligned} F \smile \overline{\nu}^{e-|F|} \in C^{\uparrow} \quad &\Leftrightarrow \quad F \smile (\nu, {}^{r(\nu)*}\nu, {}^{2r(\nu)*}\nu, \dots, {}^{(e-|F|-1)r(\nu)*}\nu) \in {}^{\bullet}C \Leftrightarrow \\ &\Leftrightarrow \quad \nu \in \{x \in {}^{r(\nu)*}\mathbb{N};\, {}^{r(\nu)*}(F \smile x \smile \overline{\nu}^{e-|F|-1^{\uparrow}}) \in {}^{\bullet}C\} = \\ &= \quad {}^{r(\nu)*}\{x \in \mathbb{N};\, F \smile x \smile \overline{\nu}^{e-|F|-1^{\uparrow}} \in {}^{\bullet}C\} =: {}^{r(\nu)*}B_{F,e}. \end{aligned}$$

Denote $B = \bigcap\{B_{F,e};\, e \in E, F \in \langle A' \rangle^{<e}\} \subseteq \mathbb{N}$ the finite intersection. We have $\nu \in {}^{r(\nu)*}B$, so B is nonempty (even infinite). Take $a_n \in B$.

Theorem

2) *Let $E = \mathbb{N} - n$ for some $0 \neq n \in \mathbb{N}$. Then the following are equivalent for every $C \subseteq \langle \mathbb{N} \rangle^{\in E}$:*

a) $(\exists A \in [\mathbb{N}]^\omega)\langle A \rangle^{\in E} \subseteq C$

b) $(\exists X \in [\mathbb{N}]^\omega)(\exists \nu \in {}^{\bullet}X - X)(\overline{\nu}^n \in C^\uparrow \,\&\, (\forall F \in \langle X \rangle^{<\omega})(F \smile \overline{\nu}^{n-1} \in C^\uparrow \rightarrow F \smile \overline{\nu}^n \in C^\uparrow))$.

Proof: a)⇒b): Take $X = A$ and $\nu \in {}^{\bullet}A - \mathbb{N}$.

b)⇒a): Similarly as before, we construct $A = \{a_0, a_1, \ldots\} \subseteq X$. The required property is

$$(\forall F \in \langle A \rangle^{<\omega})(\forall m : n - |F| \leq m \leq n)(F \smile \overline{\nu}^m \in C^\uparrow), \qquad (\#')$$

which implies $\langle A \rangle^{\in E} = \langle A \rangle^{\geq n} \subseteq C$.

$$(\forall F \in \langle A\rangle^{<\omega})(\forall m : n - |F| \leq m \leq n)(F\smile\overline{\nu}^m \in C^{\uparrow}), \qquad (\#')$$

The induction step is:

$$\begin{aligned} F\smile\overline{\nu}^m \in C^{\uparrow} \quad &\Leftrightarrow \quad F\smile(\nu, {}^{r(\nu)*}\nu, {}^{2r(\nu)*}\nu, \ldots, {}^{(m-1)r(\nu)*}\nu) \in {}^{\bullet}C \Leftrightarrow \\ &\Leftrightarrow \quad \nu \in \{x \in {}^{r(\nu)*}X; {}^{r(\nu)*}(F\smile x\smile\overline{\nu}^{m-1\uparrow}) \in {}^{\bullet}C\} = \\ &= \quad {}^{r(\nu)*}\{x \in X; F\smile x\smile\overline{\nu}^{m-1\uparrow} \in {}^{\bullet}C\} =: {}^{r(\nu)*}B'_{F,m}. \end{aligned}$$

Take $a_n \in B' = \bigcap\{B'_{F,m}; F \in \langle A'\rangle^{<\omega}, n - |F| \leq m \leq n, 0 < m\} \subseteq X$. We get

$$(\forall F \in \langle A' \cup \{a_n\}\rangle^{<\omega})(\forall m : n - |F| \leq m \leq n-1)(F\smile\overline{\nu}^m \in C^{\uparrow})$$

from where (#') follows thanks to $A' \cup \{a_n\} \subseteq X$ and the assumption on X.

Grading colorings

Furtheron, we assume $\textcircled{\kappa} > 2^\omega$.

A coloring of a set X is any family $\mathcal{C}$ of mutually disjoint sets such that $X \subseteq \bigcup \mathcal{C}$.

If $\mathcal{C}$ is a finite coloring of $X \subseteq \langle \mathbb{N} \rangle^{<\omega}$, then $\mathcal{C}^\uparrow = \{C^\uparrow; C \in \mathcal{C}\}$ is a coloring of $X^\uparrow$.

In particular, if X is some $\langle \mathbb{N} \rangle^e$ or $\langle \mathbb{N} \rangle^{<\omega}$, then $\mathcal{C}^\uparrow$ is a coloring of ${}^\cdot X$ (as ${}^\cdot X \subseteq X^\uparrow$).

Ramsey's theorem

Theorem (Ramsey)

Let $\mathcal{C}$ be a finite coloring of $\langle\mathbb{N}\rangle^e$, $e \in \mathbb{N}$. Then there is an infinite $A \subseteq \mathbb{N}$ such that $\langle A\rangle^e \subseteq C$ for some $C \in \mathcal{C}$.

We can even prove the following stronger statement directly:

Theorem

Let $E \subseteq \mathbb{N}$ be finite and let $\mathcal{C}_e$ be a finite coloring of $\langle\mathbb{N}\rangle^e$ for every $e \in E$. Then there are an infinite set $A \subseteq \mathbb{N}$ and colors $C_e \in \mathcal{C}_e$ such that $\langle A\rangle^e \subseteq C_e$ for every $e \in E$.

Proof: Let us take arbitrary $\nu \in {}^*\mathbb{N} - \mathbb{N}$ and denote $C_e^{\uparrow} \in \mathcal{C}_e^{\uparrow}$ the colors of $\overline{\nu}^e$ (that is $\overline{\nu}^e \in C_e^{\uparrow}$) for $e \in E$.

The existence of A follows directly from the nonstandard criterion of fullness 1) for $C = \bigcup_{e \in E} C_e$.

Hilbert's theorem

Theorem (Hilbert)

Let $\mathcal{C}'$ be a finite coloring of $\mathbb{N}$ and $m \in \mathbb{N}$. Then there are a finite set $A \subseteq \mathbb{N}$ with $|A| = m$, an infinite set $B \subseteq \mathbb{N}$, and some $C' \in \mathcal{C}'$ such that $b + \sum F \in C'$ for all $F \subseteq A$ and $b \in B$.

Again, we can prove even more:

Theorem

Let $\mathcal{C}'$ be a finite coloring of $\mathbb{N}$ and $m \in \mathbb{N}$. Then there are an infinite set $A \subseteq \mathbb{N}$ and $C' \in \mathcal{C}'$ such that $\sum F \in C'$ for all $F \in \langle A \rangle^{\leq m}$.

Hilbert's theorem

Theorem

Let $\mathcal{C}'$ be a finite coloring of $\mathbb{N}$ and $m \in \mathbb{N}$. Then there are an infinite set $A \subseteq \mathbb{N}$ and $C' \in \mathcal{C}'$ such that $\sum F \in C'$ for all $F \in \langle A \rangle^{\leq m}$.

Proof: We pick a $+$-idempotent $\nu \in {}^*\mathbb{N} - \mathbb{N}$ and denote C' its color (that is $\nu \in {}^{\bullet}C'$).

We obtain A from the criterion 1) for $C = \{F \in \langle \mathbb{N} \rangle^{<\omega}; \sum F \in C'\}$ and $E = \{0, \dots, m\}$.

Let us verify the condition $\overline{\nu}^e \in C^{\uparrow}$: As ν is idempotent for $\sum$, we get $\nu \sim \sum^{\uparrow} \overline{\nu}^e$ and therefore $\sum^{\uparrow} \overline{\nu}^e \in {}^{\bullet}C'$, which is equivalent to $\overline{\nu}^e \in C^{\uparrow}$ (because $(\forall \overline{\alpha})(\overline{\alpha} \in {}^{\bullet}C \leftrightarrow \sum \overline{\alpha} \in {}^{\bullet}C')$ holds true in ${}^{\bullet}\mathbb{N}$).

Hindmann's theorem

Hilbert's theorem is the maximum in this direction that can be proven using the criterion 1). But from the criterion 2) we get:

Theorem (Hindmann)

Let $\mathcal{C}$ be a finite coloring of $\mathbb{N}$. Then there are an infinite set $A \subseteq \mathbb{N}$ and $C \in \mathcal{C}$ such that $\sum F \in C$ for all $F \in \langle A \rangle^{<\omega}$.

Proof: Let us take $X = \mathbb{N}$, $\nu \in {}^*\mathbb{N} - \mathbb{N}$ a $+$-idempotent, denote C' its color (that is $\nu \in {}^\bullet C'$), and apply the criterion 2) for $C = \{F \in \langle \mathbb{N} \rangle^{<\omega}; \sum F \in C'\}$ and $n = 2$.

To verify that

$$(\forall F \in \langle X \rangle^{<\omega})(F \smile \overline{\nu}^{n-1} \in C^{\uparrow} \to F \smile \overline{\nu}^{n} \in C^{\uparrow}),$$

it is enough to observe that $\sum F + \nu \sim \sum F + \nu + {}^*\nu$ for every $F \in \langle \mathbb{N} \rangle^{<\omega}$, which follows directly from $+$-idempotence of ν.

The same proof gives also the following generalization:

For operation $g : \mathbb{N}^2 \to \mathbb{N}$, the right iteration of g is the function $G : \mathbb{N}^{<\omega} \to \mathbb{N}$ defined as $G(\emptyset) = 0$, $G(x) = x$, and $G(\overline{x}) = g(x_0, g(x_1, g(\dots g(x_{l(\overline{x})-2}, x_{l(\overline{x})-1})\dots)))$ if $l(\overline{x}) \geq 2$.

Theorem

Let $\mathcal{C}$ be a finite coloring of $\mathbb{N}$, $g : \mathbb{N}^2 \to \mathbb{N}$ such that there is a g-idempotent element in ${}^*\mathbb{N} - \mathbb{N}$, and $G : \mathbb{N}^{<\omega} \to \mathbb{N}$ be the right iteration of g. Then there are an infinite set $A \subseteq \mathbb{N}$ and $C \in \mathcal{C}$ such that $G(F) \in C$ for all $F \in \langle A \rangle^{<\omega}$.

Partition regularity

We say that a tuple $\overline{x}$ is injective if its elements are mutually distinct.

A set $X \subseteq \mathbb{N}^n$ is called [injectively] partition regular if for every finite coloring $\mathcal{C}$ of $\mathbb{N}$ there is an [injective] $\overline{x} \in X$ that is monochromatic (i.e. $\overline{x} \in C$ for some $C \in \mathcal{C}$).

An equation $f(\overline{x}) = 0$ over $\mathbb{N}$ is called [injectively] partition regular if the set of all its solutions is.

Proposition

Let $X \subseteq \mathbb{N}^n$. The following statements are equivalent:

1) *X is [injectively] partition regular.*
2) *There is [injective] $\overline{\nu} \in {}^*X$ such that $\nu_0 \sim \cdots \sim \nu_{n-1}$.*
3) *There is [injective] $\overline{\nu} \in {}^{\bullet}X$ such that $\nu_0 \sim \cdots \sim \nu_{n-1}$.*

Partition regularity of $x + y = 2z$

Let us apply this to the question of partition regularity of the equation $x + y = 2z$.

Proposition

The equation $x + y = 2z$ over $\mathbb{N}$ is injectively partition regular.

Proof: Let $\nu \in {}^*\mathbb{N}$ be a $+$-idempotent. We set

$$\begin{aligned} \alpha &= 2\nu + {}^{**}\nu, \\ \beta &= 2\nu + 2{}^*\nu + {}^{**}\nu, \\ \gamma &= 2\nu + {}^*\nu + {}^{**}\nu. \end{aligned}$$

It is easy to compute that $\alpha + \beta = 2\gamma$.

Now each of α, β, γ can be written in a form $2\delta +^{\uparrow} \varepsilon$ with $\delta \sim \varepsilon \sim \nu$:

$$\begin{aligned} \alpha &= 2\nu +^{\uparrow} {}^*\nu, \\ \beta &= 2(\nu + {}^*\nu) +^{\uparrow} \nu, \\ \gamma &= 2\nu +^{\uparrow} (\nu + {}^*\nu). \end{aligned}$$

Then we get $\alpha \sim \beta \sim \gamma$.

van der Waerden's Theorem for length 3

Because any solution x, y, z of the above equation is an arithmetic progression (ordered x, z, y or y, z, x) of length 3, we get:

Corollary (van der Waerden's Theorem for length 3)

Let $\mathcal{C}$ be a finite coloring of $\mathbb{N}$. Then there is $C \in \mathcal{C}$ and an arithmetic progression $a, a+d, a+2d \in C$.

Thanks

Thank you for your attention.