

# (Un)provability of Fermat's last theorem and Catalan's conjecture in formal arithmetics

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## Section 1

# (Un)provability of Fermat's Last Theorem

# Fermat's Last theorem

## Theorem (FLT, Wiles, 1995)

For  $n > 2$  the equation

$$x^n + y^n = z^n$$

has no solution  $x, y, z \neq 0$  in  $\mathbb{N}$ .

The original Wiles's proof is **not** done in ZFC. It uses existence of Grothendieck's universes which is equivalent to existence of (strongly) inaccessible cardinals.

Nevertheless, it is believed that much less is used in principle.

## Provability of FLT

- McLarty, 2011-12: FLT is provable in **ZFC**, even in **finite order arithmetic** [*C. McLarty, The large structures of Grothendieck founded on finite order arithmetic, arXiv:1102.1773v4*] and partially in **second order arithmetic** [*C. McLarty, Zariski cohomology in second order arithmetic, arXiv:1207.0276v2*] (use of Grothendieck's universes can be eliminated in the Wiles's proof)
- Macintyre, 2011: A (quite detailed) sketch of a project of proving FLT in **PA** (according the same lines as Wiles's proof but not a routine translation) [*A. Macintyre, The impact of Gödel's incompleteness theorems on mathematics – Appendix, Kurt Gödel and the Foundations of Mathematics: Horizons of Truth, Cambridge University Press, Cambridge, 2011.*]
- Smith, 1992: FLT for some small even values of exponent (e.g.  $n = 4, 6, 10$ ) is provable in  **$IE_1$**  (= **bounded existential induction**) [*S. T. Smith, Fermat's last theorem and Bezout's theorem in GCD domains, J. Pure Appl. Alg. 79 (1992), 63–85.*]

## Unprovability of FLT

- Shepherdson, 1964: FLT for  $n = 3$  is not provable in **IOpen (open induction)** [*J. C. Shepherdson, A nonstandard model for a free variable fragment of number theory, Bull. 1'Acad. Pol. Sci. 12 (1964), 79–86.*]
- Kołodziejczyk, 2011: FLT for  $n = 3$  is not provable in  **$T_2^0$  (sharply bounded induction – bounded by length of term)** [*L. A. Kołodziejczyk, Independence results for variants of sharply bounded induction, Ann. Pure Appl. Logic 162 (2011), 981–990.*]

## Section 2

# Exponential arithmetics

We will be working with models  $\langle \mathcal{B}, e \rangle$  where  $\mathcal{B} \models I\Sigma_1$  is a background model and  $e$  an exponential satisfying axioms **Exp**:

(e0) " $e : B \times A \rightarrow B$  for some substructure  $\mathcal{A}$  of  $\mathcal{B}$  with  $\mathcal{A} \models \text{Pr}$ "

(e1)  $(x = 1 \vee y = 0) \leftrightarrow e(x, y) = 1$ ,

(e2)  $x \neq 0 \rightarrow e(x, y) \neq 0$ ,

(e3)  $e(x, 1) = x$ ,

(e4)  $e(x, y + z) = e(x, y) \cdot e(x, z)$ ,

(e5)  $e(\prod_{i < l} x_i, y) = \prod_{i < l} e(x_i, y)$  (right hand side is correct thanks to (e7)),

(e6)  $e(e(x, y), z) = e(x, yz)$ ,

(e7) "for any  $b \in B$ , the set  $\{(x, e(x, y)); x < b\}$  is coded in  $B$ ",

whenever  $y, z \in A$ ,  $x \in B$  and  $(x_i)_{i < l}$  is a sequence coded in  $B$  of length  $l \in B$ .

Note that in  $\mathcal{B}$  the usual exponential  $x^y$  is definable. In general,  $e$  differs from  $x^y$  (although it follows from Exp that  $e(m, n) = m^n$  for  $m, n \in \mathbb{N}$ ).

Besides FLT, we will be also interested in the Catalan's conjecture:

Theorem (Catalan's conjecture, Mihăilescu, 2004)

*The only solution of*

$$e(a, n) - e(b, m) = 1$$

*in  $\mathbb{N}$  with  $a, b, m, n > 1$  is  $a = m = 3, b = n = 2$ .*

Let us also recall the statement of the ABC conjecture:

Conjecture (ABC, Mochizuki, 2012???)

For every  $\varepsilon > 0$  there is  $K_\varepsilon$  such that for all coprime  $a, b, c$  with  $a + b = c$  we have  $c < K_\varepsilon \text{rad}(abc)^{1+\varepsilon}$ ,

where  $\text{rad}(x)$  is the product of all different primes dividing  $x$ .

## Results

We will prove:

- $Th(\mathbb{N}) + Exp \not\vdash FLT$  (moreover, FLT can be violated by unboundedly many exponents  $n$  and, independently on  $n$ , by unboundedly many pairwise linearly independent triples  $x, y, z$ ),
- (assuming ABC conjecture in  $\mathbb{N}$ )  $Th(\mathbb{N}) + Exp \vdash \text{Catalan's conjecture}$  (moreover, Exp can be replaced here just by axioms (e0)–(e4)),
- (assuming ABC conjecture in  $\mathbb{N}$ )  $Th(\mathbb{N}) + Exp + (e8) \vdash FLT$ , where (e8) is “If  $x$  and  $y$  are coprime, then so are  $e(x, a)$  and  $e(y, b)$ ” (moreover, Exp can be replaced here by (e0)–(e4) and (e5’), which is a finite variant of (e5)).

## Section 3

# Construction of exponentials

Let  $\mathcal{B} \models I\Sigma_1$  be fixed and assume we have an exponential  $e : B \times A \rightarrow B$  satisfying Exp.

Then by (e5) the values of  $e$  are uniquely determined by values  $e(q, y)$  for  $q$  primes in  $\mathcal{B}$ . Moreover, by (e7),

$$e(q, y) = \prod_{p \in \mathbb{P}} p^{\varepsilon(y)_{pq}}$$

So  $e$  is completely determined by the matrices  $\varepsilon(y)_{pq}$  where  $p, q$  are prime numbers in  $\mathcal{B}$  and  $y \in A$ .

Moreover, by (e4) and (e6),  $\varepsilon : y \mapsto \varepsilon(y)_{pq}$  is a semiring homomorphism from  $\mathcal{A}$  to the ring  $M_{\mathbb{P}}^{\text{good}}(\mathcal{B})$  of all good  $\mathbb{P} \times \mathbb{P}$ -matrices over  $\mathcal{B}$ .

A matrix  $M$  is good if for any  $J \in B$  there is  $I = I_M(J) \in B$  such that

- i) all non-zero values  $M_{ij}$  from first  $J$  columns are in the first  $I$  rows,
- ii) the restricted matrix  $(M_{ij})_{i < I, j < J}$  is coded in  $\mathcal{B}$ .

On the other hand if a semiring homomorphism  $\varepsilon : A \rightarrow M_{\mathbb{P}}^{\text{good}}(\mathcal{B})$  is given, then we can define an exponential  $e$  by:

$$\begin{aligned}e(0, 0) &= 1, \\e(0, z) &= 0, \\e(x, y) &= v^{-1}(\varepsilon(y)v(x)),\end{aligned}$$

where  $v : x \mapsto (v_p(x))_{p \in \mathbb{P}}$  is the usual (additive p-adic) valuation in  $\mathcal{B}$ .

In fact, there is a **bijection between these semiring homomorphisms and exponentials**:

### Proposition

Let  $\mathcal{B} \models \text{I}\Sigma_1$  and  $\mathcal{A} \subseteq \mathcal{B}$ . Then the maps  $e \mapsto \varepsilon^e$  and  $\varepsilon \mapsto e^\varepsilon$  defined above, are mutual inverses and the following are equivalent:

- The exponential  $e = e^\varepsilon : B \times A \rightarrow B$  satisfies Exp.
- The map  $\varepsilon = \varepsilon^e : A \rightarrow M_{\mathbb{P}}^{\text{good}}(\mathcal{B})$  is a semiring homomorphism.

## Examples of exponentials

- Let  $\mathcal{A} = \mathcal{B}$  and  $\varepsilon(y) = yI$ , for  $y \in B$ , where  $I$  is the identity matrix. Then  $e(x, y) = x^y$  (the original exponential in  $\mathcal{B}$ ).
- Let  $\mathcal{A} = \mathcal{B}$ ,  $f$  an automorphism of  $\mathcal{B}$  and  $\varepsilon(y) = f(y)I$ , for  $y \in B$ . Then  $e(x, y) = x^{f(y)}$ .
- An exponential  $e$  satisfies (e8)  $\Leftrightarrow$  all matrices  $\varepsilon(y)$  are diagonal  $\Leftrightarrow$   $e$  is of the form  $e_f(\prod_i p_i^{e_i}, a) = \prod_i p_i^{e_i f_{p_i}(a)}$  with  $f = (f_p; p \in \mathbb{P})$  homomorphisms from  $\mathcal{A}$  to  $\mathcal{B}$ .

## Non-diagonal example

Suppose that  $\mathcal{A} \subseteq \mathcal{B}$  is a model of Pr and that every  $\mathbb{Z}$ -component of  $\mathcal{A}$  contains an element  $O$  divisible by all  $n \in \mathbb{N}$ .

Denote  $\mathcal{O}$  the set of all such elements  $O$ . Easily,  $\mathcal{O}$  is closed under  $+$ ,  $-$ ,  $\cdot$  and contains  $0$ .

Then any  $(0, +, -, \cdot)$ -homomorphism  $\varepsilon : \mathcal{O} \rightarrow M_{\mathbb{P}}^{\text{good}}(\mathcal{B})$  can be easily extended to a semiring homomorphism  $\varepsilon : \mathcal{A} \rightarrow M_{\mathbb{P}}^{\text{good}}(\mathcal{B})$  (by setting  $\varepsilon(O + n) = \varepsilon(O) + nI$ , where  $O \in \mathcal{O}$  and  $n \in \mathbb{Z}$ ).

**Example:**

$$\varepsilon : \mathcal{O} \mapsto \begin{pmatrix} O/n & O/n & \cdots & O/n & 0 & \cdots & \\ \vdots & \vdots & \ddots & \vdots & 0 & \cdots & \\ O/n & O/n & \cdots & O/n & 0 & \cdots & \\ 0 & 0 & \cdots & 0 & O & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \end{pmatrix}$$

## Section 4

### Results and proofs

## Theorem

- 1) *There is a model  $\langle \mathcal{B}, e \rangle \models \text{Th}(\mathbb{N}) + \text{Exp}$  containing an unbounded set  $E \subseteq B$  of exponents and (in every coordinate) unbounded set  $T \subseteq B^3$  of pairwise linearly independent triples  $(a, b, c)$  such that for every  $n \in E$  and  $(a, b, c) \in T$  we have*

$$e(a, n) + e(b, n) = e(c, n).$$

Moreover:

- *For any fixed  $y$ ,  $e(x, y)$  is a definable function of  $x$  in  $\mathcal{B}$ .*
  - *$e$  is definable in the expansion  $\langle \mathcal{B}, \mathcal{N} \rangle$  of  $\mathcal{B}$  by a predicate  $\mathcal{N}(x)$  expressing “ $x$  is a standard number”.*
- 2) *There is a substructure  $\langle \mathcal{A}, e \rangle \subseteq \langle \mathcal{B}, e \rangle$  with  $e$  total and  $\mathcal{A} \models \text{Pr}$  such that  $E \subseteq A$ ,  $T \subseteq A^3$ . (Thus, in addition to axioms of  $\text{Pr}$ ,  $\langle \mathcal{A}, e \rangle$  satisfies all quantifier-free statements true in  $\langle \mathcal{B}, e \rangle$ .)*

**Proof:** See the board.

Let  $S$  be a theory (in the language of arithmetic  $\langle 0, 1, +, \cdot, \leq \rangle$ ) stronger than  $I\Sigma_1$  such that, for some  $K \in \mathbb{N}$ ,  $S$  proves (“ $a, b, c$  coprime” &  $a + b = c$ )  $\rightarrow c < K \text{rad}(abc)^{1+1/3}$ , and the Catalan conjecture (using the exponential  $x^y$  definable in  $S$ ). By Mochizuki's (?) and Mihăilescu's results, we may take  $S = Th(\mathbb{N})$ .

We denote by  $Exp'$  the axioms (e0)–(e4).

### Theorem

*Let  $S$  be as above. Catalan Conjecture for  $e$  is provable in  $S + Exp'$ .*

**Proof:** See the board.

Recall

(e8) "If  $x$  and  $y$  are coprime, then so are  $e(x, a)$  and  $e(y, b)$ ."

This is equivalent to all corresponding matrices  $\varepsilon(a)$  being diagonal.  
Note also that (e8) is still much weaker than induction for  $e$ .

We denote the finite version of (e5) by

(e5')  $e(xy, z) = e(x, z) \cdot e(y, z)$

Let  $T$  be a theory (in the language of arithmetic  $\langle 0, 1, +, \cdot, \leq \rangle$ ) stronger than  $I\Sigma_1$  such that, for some  $K \in \mathbb{N}$  and some  $\varepsilon > 0$ ,  $T$  proves (" $a, b, c$  coprime" &  $a + b = c$ )  $\rightarrow c < K \text{rad}(abc)^{1+\varepsilon}$ , and the Fermat's Last Theorem (using the exponential  $x^y$  definable in  $T$ ). We may again take  $T = Th(\mathbb{N})$ .

### Theorem

*Let  $T$  be a theory as above. Fermat's Last Theorem for  $e$  is provable in  $T + \text{Exp}' + (e5') + (e8)$ .*

**Proof:** Analogous to the proof of Catalan's conjecture.

## Open Questions

### Open Problem

For which arithmetical theories  $S$  does there exist a model  $\langle \mathcal{B}, e \rangle \models S + Exp + \text{“}e \text{ is total”}$  such that Fermat's Last Theorem for  $e$  does not hold in  $\langle \mathcal{B}, e \rangle$ ? In particular, is there such a model for  $S = Th(\mathbb{N})$ ?

### Open Problem

Is there a model  $\mathcal{B} \models Th(\mathbb{N})$  (or at least of  $I\Sigma_1$ ) that permits a semiring homomorphism  $\varepsilon : B \rightarrow M_{\mathbb{P}}^{good}(\mathcal{B})$  with some values  $\varepsilon(b)$  non-diagonal?

**Thank you.**

*[P. Glivický and V. Kala, Fermat's last theorem and Catalan's conjecture in weak exponential arithmetics, to appear in Mathematical Logic Quarterly, arXiv: 1602.03580]*